Optimization of Economic Production Quantity and Profits under Markovian Demand

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Abstract— In most manufacturing industries; demand uncertainty affects effective planning and control of production levels that maximize profits. In this paper, a new mathematical model is developed to optimize economic production quantity (EPQ) of a single-item, finite horizon, periodic review inventory problem with Markovian demand. In this model, sales price and inventory replenishment periods are uniformly fixed over the planning horizon. Adopting a Markov decision process approach, the states of a Markov chain represent possible states of demand for the inventory item. The production cost, holding cost, shortage cost and sales price are combined with demand and inventory positions to generate the profit for the decision problem. The objective is to determine in each period of the planning horizon an optimal economic production quantity so that the long run profits are maximized for a given state of demand. Using weekly equal intervals, the decisions of how much to produce are made using dynamic programming over a finite period planning horizon. A numerical example demonstrates the existence of an optimal state-dependent economic production quantity as well as the corresponding profits of item.

Keywords—*Economic Production Quantity, Markovian demand, Profits*

1. INTRODUCTION

In production-inventory management, the zeal for manufacturing industries to plan for optimal production levels that sustain random demand leaves a lot to be desired. In practice, when production exceeds quantity demanded, inventory carrying costs accumulate which affect profit margins of the manufacturer. Similarly, production levels below demand impose shortage costs and loss of good will from potential customers. Both cases drastically reduce profit margins unless proper planning and coordination is put in place to establish optimal production levels in a given manufacturing industry. In an effort to achieve this goal, two major problems are usually encountered:

(i)Determining the economic production quantity (EPQ) of the item in question
(ii)Determining the optimal profits associated with the economic production quantity(EPQ) when demand is uncertain.
In this paper, a production-inventory system is considered whose goal is to optimize the economic production quantity (EPQ) and profits associated with producing and holding inventory of an item. At the beginning of each period, a major has to be made: namely whether to produce additional units of the item in inventory or cancel production and utilize the available units in inventory. In either case, the economic production quantity (EPQ) must be determined that optimize profits in a given production-inventory system.

According to Goya[1],the EPQ may be computed each time a product is scheduled for production when all products are produced on a single machine. The distribution of lot size can be computed when the demand for each product is a stochastic variable with a known distribution. The distribution is independent for non-overlapping time periods and identical for equal time epochs. In related work by Khouja[2],the EPQ is shown to be determined under conditions of increasing demand, shortages and partial backlogging. In this case the unit production cost is a function of the production process and the quality of the production process deteriorates with increased production rate.Tabucanno and Mario [3] also presented a production order quantity model with stochastic demand for a chocolate milk manufacturer. In this paper, production planning is made complex by the stochastic nature of demand for the manufacturer's product. An analysis of production planning via dynamic programming approach shows that the company's production order variable is convex.
In the article presented by Kampf and Kochel[4], the stochastic lot sizing problem is examined where the cost of waiting and lost demand is taken into consideration. In this model, production planning is made possible by making a trade-off between the cost of waiting and the lost demand.

The four models presented have some interesting insights regarding economic production quantity in terms of optimization and stochastic demand. However, profit maximization is not a salient factor in the optimization models cited.

2. MODEL FORMULATION

2.1 Notation and Assumptions

i,j = States of demand
N^{Z}_{ij} = Number of customers
D^{Z}_{ij} = Quantity demanded
F = Favorable state
U = Unfavorable state
I^{Z}_{ij} = Quantity in inventory
n,N = Stages
P^{Z}_{ij} = Profits
Z = Production lot sizing policy
V^{Z}_{i} = Expected profits
N^{Z} = Customer matrix
a^{Z}_{i} = Accumulated profits
D^{Z} = Demand matrix
c_p = Unit production cost
I^{Z} = Inventory matrix

c_h = Unit holding cost
P^Z = Profit matrix
c_s = Unit shortage cost
Q^Z=Demand transition matrix
Q^Z_{ij}= Demand transition probability
p_s = Unit sales price
E^Z_i = Economic Production Quantity
i,j ϵ {F,U} Z ϵ {0,1} n=1,2,…………….N

We consider a production-inventory system of a single item whose demand during a chosen period over a fixed planning horizon is classified as either *Favorable* (state F) or *Unfavorable* (state U) and the demand of any such period is assumed to depend on the demand of the preceding period. The transition probabilities over the planning horizon from one demand state to another may be described by means on a Markov chain. Suppose one is interested in determining the optimal course of action, namely to produce additional units of item (a decision denoted by Z=1) or not to produce additional units (a decision denoted by Z=0) during each time period over the planning horizon where Z is a binary decision variable. Optimality is defined such that the maximum expected profits are accumulated at the end of N consecutive time periods spanning the planning horizon. In this paper, a two-period (N=2) planning horizon is considered.

2.2 Finite dynamic programming formulation

Recalling that the demand can be in state F or in state U, the problem of finding an optimal economic production quantity may be expressed as a finite period dynamic programming model.

Let $g_n(i)$ denote the expected total profits accumulated during periods *n,n+1,........,N* given that the state of the system at the beginning of period *n* is iε{F,U}.The recursive equation relation g_n and g_{n+1} is

$$g_n(i) = max_Z[Q^Z_{iF}(P^Z_{iF} + g_{n+1}(F)), Q^Z_{iU}(P^Z_{iU} + g_{n+1}(U))] \quad (1)$$

iε{F,U} ,*n=1,2,................................N* together with the final conditions $g_{N+1}(F) = g_{N+1}(U) = 0$

This recursive relationship may be justified by noting that the cumulative total profits $P^Z_{ij}+ g_{n+1}(j)$ resulting from reaching jε{F,U }at the start of period *n+1* from state iε{F,U}at the start of period *n* occurs with probability Q^Z_{ij}.

Clearly, $V^Z = Q^Z (P^Z)^T$, Zε {0,1} (2)
where "T" denotes matrix transposition, and hence the dynamic programming recursive equations

$$g_n(i) = max_Z[v^Z_i + Q^Z_{iF}\, g_{n+1}(F) + Q^Z_{iU}\, g_{n+1}(U)] \quad (3)$$

iε{F,U} n=1,2,...........N-1, Zε{0,1}.

$$g_N(i) = max\ [\ v^Z_i] \quad iε\{F,U\} \quad (4)$$

result where (4) represents the Markov chain stable state.

3. COMPUTING Q^Z, P^Z and E^Z

The demand transition probability from state i ε{F, U}to state jε{F, U } given production lot sizing policy Zε {0,1} may be taken as the number of customers observed when demand is initially in state i and later with demand changing to state j, divided by the number of customers over all states.

$$Q^Z_{ij} = N^Z_{ij}/[N^Z_{iF} + N^Z_{iU}] \quad (5)$$

iε{F,U} , Zε {0,1}
When demand outweighs on-hand inventory, the profit matrix P^Z may be computed by the relation:

$$P^Z = p_s[D^Z] - (c_p + c_h + c_s)(D^Z - I^Z)$$

Therefore

$$P^Z_{ij} = \begin{cases} p_s D^Z_{ij} - (c_p + c_h + c_s)(D^Z_{ij} - I^Z_{ij}) & if\ \ D^Z_{ij} > I^Z_{ij} \\ p_s D^Z_{ij} - c_h I^Z_{ij} & if\ \ D^Z_{ij} \le I^Z_{ij} \end{cases} \quad (6)$$

for all i,j ε{F,U}, Zε {0,1}
A justification for expression (6) is that D^Z_{ij} -I^Z_{ij} units must be produced in order to meet the excess demand. Otherwise production is cancelled when demand is less than or equal to on-hand inventory. The economic production quantity when demand is initially in state i ε{F,U},given production lot sizing policy Zε {0,1} is

$$E^Z_i = (D^Z_{iF} - I^Z_{iF}) + (D^Z_{iU} - I^Z_{iU})$$

iε{F,U}, Zε {0,1} (7)

The following conditions must however hold:

1. $E^Z_i > 0$ when $D^Z_{ij} > I^Z_{ij}$ and $D^Z_{ij} = 0$ when $D^Z_{ij} \le I^Z_{ij}$
2. Z=1 when $c_p > 0$, and Z=0 when $c_p = 0$
3. $c_s > 0$ when shortages are allowed and $c_s = 0$ when shortages are not allowed.

4. COMPUTING AN OPTIMAL ECONOMIC PRODUCTION QUANTITY

The optimal EPQ is found in this section for each time period separately,

4.1 Optimization during period 1

When demand is Favourable (i.e. in state F), the optimal production lot sizing decision during period 1 is

$$Z = \begin{cases} 1 & if \quad v^1_F > v^0_F \\ 0 & if \quad v^1_F \le v^0_F \end{cases}$$

The associated total profits and EPQ are then

$$g_1(F) = \begin{cases} v^1_F & if \quad Z = 1 \\ v^0_F & if \quad Z = 0 \end{cases}$$

and

$$E^Z_F = \begin{cases} (D^1_{FF} - I^1_{FF}) + (D^1_{FU} - I^1_{FU}) & if\ \ Z = 1 \\ 0 \quad if \quad Z = 0 \end{cases}$$

respectively. Similarly, when demand is Unfavorable (i.e. in state U), the optimal production lot sizing policy during period 1 is

$$Z = \begin{cases} 1 & if \quad v^1_U > v^0_U \\ 0 & if \quad v^1_U \le v^0_U \end{cases}$$

In this case, the associated total profits and EPQ are

$$g_1(U) = \begin{cases} v_U^1 & if \quad Z = 1 \\ v_U^0 & if \quad Z = 0 \end{cases}$$

and

$$E_U^Z = \begin{cases} (D_{UF}^1 - I_{UF}^1) + (D_{UU}^1 - I_{UU}^1) & if \;\; Z = 1 \\ 0 \;\; if \quad Z = 0 \end{cases}$$

respectively.

4.2 Optimization during period 2

Using dynamic programming recursive equation (1), and recalling that a^Z_i denotes the already accumulated profits at the end of period 1 as a result of decisions made during that period,it follows that:

$$a_i^Z = v_i^Z + Q_{iF}^Z max[v_F^1, v_F^0] + Q_{iU}^Z max[v_U^1, v_U^0]$$
$$= v_i^Z + Q_{iF}^Z\, g_1(F) + Q_{iU}^Z\, g_1(U)$$

Therefore when demand is favorable (i.e. in state F), the optimal production lot sizing decision during period 2 is

$$Z = \begin{cases} 1 & if \;\; a_F^1 > a_F^0 \\ 0 & if \;\; a_F^1 \le a_F^0 \end{cases}$$

while the associated total profits and EPQ are

$$g_2(F) = \begin{cases} a_F^1 & if \;\; Z = 1 \\ a_F^0 & if \;\; Z = 0 \end{cases}$$

and

$$E_F^Z = \begin{cases} (D_{FF}^1 - I_{FF}^1) + (D_{FU}^1 - I_{FU}^1) & if \;\; Z = 1 \\ 0 \;\; if \quad Z = 0 \end{cases}$$

Similarly, when demand is Unfavorable (i.e. in state U), the optimal production lot sizing policy during period 2 is

$$Z = \begin{cases} 1 & if \;\; a_U^1 > a_U^0 \\ 0 & if \;\; a_U^1 \leqslant a_U^0 \end{cases}$$

while the associated total profits and EPQ are

$$g_2(U) = \begin{cases} a_U^1 & if \;\; Z = 1 \\ a_U^0 & if \;\; Z = 0 \end{cases}$$

and

$$E_U^Z = \begin{cases} (D_{UF}^1 - I_{UF}^1) + (D_{UU}^1 - I_{UU}^1) & if \;\; Z = 1 \\ 0 \;\; if \quad Z = 0 \end{cases}$$

5. CASE STUDY

In order to demonstrate the use of the model in §2-4, a real case application from *Vita foam* , a manufacturer of mattresses in Uganda is presented in this section. The demand for mattresses fluctuates from month to month. The company wants to avoid mover-producing when demand is low or under-producing when demand is high, and hence seeks decision support in terms of an optimal production lot sizing policy , the associated profits and specifically, a recommendation as to the EPQ of mattresses over a two-week period.

5.1 Data Collection

A sample of 60 customers was used. Past data revealed the following demand pattern and inventory levels over the state transitions for twelve weeks in Tables 1 and 2 below:

Table 1

Customers, Demand and Inventory Levels at state-transitions over twelve weeks for Production Lot sizing Policy 1(Z = 1)

State Transition (i,j)	Customers N^1_{ij}	Demand D^1_{ij}	Inventory I^1_{ij}
FF	40	80	74
FU	20	20	60
UF	10	120	60
UU	50	40	10

Table 2

Customers, Demand and Inventory Levels at state-transitions over twelve weeks for Production Lot sizing Policy 0(Z=0)

State Transition (i,j)	Customers N^0_{ij}	Demand D^0_{ij}	Inventory I^0_{ij}
FF	40	80	74
FU	20	20	60
UF	10	120	60
UU	50	40	10

When additional mattresses are produced (Z=1) during week 1,

$$N^1 = \begin{bmatrix} 40 & 20 \\ 10 & 50 \end{bmatrix} \quad D^1 = \begin{bmatrix} 80 & 20 \\ 120 & 40 \end{bmatrix} I^1 = \begin{bmatrix} 74 & 60 \\ 60 & 10 \end{bmatrix}$$

while if additional mattresses are *not* produced during week 1, these matrices are

$$N^0 = \begin{bmatrix} 30 & 30 \\ 20 & 40 \end{bmatrix} \quad D^0 = \begin{bmatrix} 50 & 30 \\ 160 & 80 \end{bmatrix} I^0 = \begin{bmatrix} 20 & 40 \\ 80 & 20 \end{bmatrix}$$

In either case, the unit sales price(p_s) is $20.00,the unit production cost (c_p) is $15.00,the unit holding cost per week(c_h) is $0.50,and the unit shortage cost per week(c_s) is $10.00

5.2 Computation of model parameters

Using (5) and (6), the state transition matrix and the profit matrix (in thousand dollars) for week 1 are

$$Q^1 = \begin{bmatrix} 0.67 & 0.33 \\ 0.17 & 0.83 \end{bmatrix} \qquad P^1 = \begin{bmatrix} 1.447 & 1.570 \\ 0.870 & 0.035 \end{bmatrix}$$

for the case where additional mattresses are produced during week 1, while these matrices are given by

$$Q^0 = \begin{bmatrix} 0.50 & 0.50 \\ 0.33 & 0.67 \end{bmatrix} \qquad P^0 = \begin{bmatrix} 0.235 & 0.580 \\ 1.160 & 0.070 \end{bmatrix}$$

for the case when additional mattresses are *not* produced during week 1.

When additional mattresses are produced (Z=1), the matrices Q^1 and P^1 yield the profits (in thousand dollars)

$$v_F^1 = (0.67)(1.447) + (0.33)(1.570) = 1.488$$
$$v_U^1 = (0.17)(0.870) + (0.83)(0.035) = 0.177$$

However, when additional mattresses are not produced (Z=0), the matrices Q^0 and P^0 yield the profits (in US dollars)

$$v_F^0 = (0.50)(0.235) + (0.50)(0.580) = 0.408$$
$$v_U^0 = (0.33)(1.160) + (0.67)(0.07) = 0.430$$

5.2 The optimal production lot sizing policy and EPQ

Since 1.488 > 0.408, it follows that Z=1 is an optimal production lot sizing policy for week 1 with associated total profits of $1.488 and an EPQ of 80-74 = 6 units when demand is favorable. Since 0.430 > 0.177, it follows that Z = 0 is an optimal production lot sizing policy for week 1 with associated total profits of $0.430 and an EPQ of 0 units when demand is unfavorable.

If demand is favorable, then the accumulated profits at the end of week 1 are

$$a_F^1 = 1.488 + (0.67)(1.488) + (0.33)(0.430) = 2.447$$
$$a_F^0 = 0.408 + (0.50)(1.488) + (0.50)(0.430) = 1.367$$

Since 2.447 > 1.367, it follows that Z = 1 is an optimal production lot sizing policy for week 1 with associated accumulated profits of $ 2.447 and an EPQ of 80 -74 = 6 units for the case of favorable demand.

However, if demand is unfavorable, then the accumulated profits at the end of week 1 are

$$a_U^1 = 0.177 + (0.17)(1.488) + (0.83)(0.430) = 0.786$$
$$a_U^0 = 0.430 + (0.33)(1.488) + (0.67)(0.430) = 1.209$$

Since 1.209 > 0.786, it follows that Z=0 is an optimal production lot sizing policy for week 2 with associated accumulated profits of $1.209 and an EPQ of 0 units for the case of unfavorable demand.

When shortages are not allowed, the values of Z and $g_n(i)$ and E^Z_i may be computed for i ε{F, U}in a similar fashion after substituting $c_s = 0$ the matrix function

$P^S = p_s[D^Z] – (c_p + c_h + c_s)[D^Z – I^Z]$

6. CONCLUSION

A production -inventory model was presented in this paper. The model determines an optimal production lot sizing policy, profits and the EPQ of a given item with stochastic demand. The decision of whether or not to produce additional inventory units is modeled as a multi-period decision problem using dynamic programming over a finite period planning horizon. The working of the model was demonstrated by means of a real case study.

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