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Structure and Study of Elements in Ternary Γ-Semigroups

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Abstract: In this paper we introduce the notion of a ternary Γ-semigroup is introduced and some examples are given. Further the terms commutative ternary Γ-semigroup, quasi commutative ternary Γ-semigroup, normal ternary Γ-semigroup, left pseudo commutative ternary Γ-semigroup, right pseudo commutative ternary Γ-semigroup are introduced and characterized them.

In section 2, the terms; ternary Γ-subsemigroup, ternary Γ-subsemigroup generated by a subset, cyclic ternary Γ-subsemigroup of a ternary Γ-semigroup and cyclic ternary Γ-semigroup are introduced and characterized them

In section 3, we discussed some special elements of a ternary Γ-semigroups and characterized them.

INTRODUCTION

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, and the like. The theory of ternary algebraic systems was introduced by LEHMER in 1932, but earlier such structures was studied by KASNER who give the idea of n-ary algebras. LEHMER investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. Ternary semigroups are universal algebras with one associative ternary operation. The notion of ternary semigroup was known to BANACH who is credited with example of a ternary semigroup which can not reduce to a semigroup. A. Anjaneyulu [i] introduced the study of pseudo symmetric ideals in semigroups D. Madhusudhana Rao and A. Anjaneyulu [ii, iii] studied about Γ-semigroups. Further D. Madhusudhana Rao and A .Anjaneyulu and Y. Sarla [x] extended the same results to ternary semigroups. Madhusudhana Rao and Srinivasa Rao [v, vi, vii] studied about ternary semirings. In this paper mainly we extended the same results to ternary Γ-semigroups.

1: <u>TERNARY Γ-SEMIGROUPS</u>

We now introduce the notion of ternary Γ-semigroup

Definition 1.1 : Let T and Γ be two non-empty set. Then T is said to be a ***Ternary Γ-semigroup*** if there exist a mapping from T×Γ×T×Γ×T to T which maps ($x_1, \alpha, x_2, \beta, x_3$) → $[x_1\alpha x_2\beta x_3]$ satisfying the condition :

$$[[x_1\alpha x_2\beta x_3]\gamma x_4\delta x_5] = [x_1\alpha[x_2\beta x_3\gamma x_4]\delta x_5]$$
$$= [x_1\alpha x_2\beta[x_3\gamma x_4\delta x_5]]$$

$\forall\ x_i \in$ T, $1 \le i \le 5$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Note 1.2 : For the convenience we write $x_1\alpha x_2\beta x_3$ instead of $[x_1\alpha x_2\beta x_3]$

Note 1.3 : Let T be a ternary Γ-semigroup. If A, B and C are three subsets of T, we shall denote the set AΓBΓC = $\{a\alpha b\beta c : a \in A, b \in B, c \in C, \alpha, \beta \in \Gamma\}$.

Note 1.4: Any Γ-semigroup can be reduced to a ternary Γ-semigroup.

In the following some examples of ternary Γ-semigroup are given.

Example 1.5: Let T =
$$\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$$ and
Γ= $\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$. Then T is a ternary Γ-semigroup under matrix multiplication.

Example 1.6: The set Z of all integers and Γ be the set of all even integers. If $a\alpha b\beta c$ denote as usual multiplication of integers for $a, b, c \in T$ and $\alpha, \beta \in \Gamma$, then T is a ternary Γ-semigroup.

Example 1.7 : Let the set Z of all negative integers and Γ of all odd integers is a ternary Γ-semigroup but not a Γ-semigroup under the multiplication of integers.

Example: **1.8** : Let T = { 5*n* + 4 : *n* is a positive integer} and Γ = { 5*n* + 1 : n is a positive integer}. Then T is a ternary Γ-semigroup with the operation defined by $a\alpha b\beta c = a + \alpha + b + \beta + c$ where *a*, *b*, *c* ∈ S, $\alpha, \beta \in \Gamma$ and + is the usual addition of integers.

Example: **1.9** : Let T = { 4*n* + 1 : *n* is a positive integer} and Γ = { 4*n* + 3 : n is a positive integer}. Then T is a ternary Γ-semigroup with the operation defined by $a\alpha b\beta c = a + \alpha + b + \beta + c$ where *a*, *b*, *c* ∈ S, $\alpha, \beta \in \Gamma$ and + is the usual addition of integers.

Example 1.10 : Let T = $\{a, b, c\}$ and T = Γ such that $a\alpha b\beta c$ = $(x*y)*z$ for all *x, y, z* ∈ T and α, β ∈ Γ where * is defined by the table

*	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*b*
c	*a*	*c*	*c*

Then T is a ternary Γ-semigroup.

Example 1.11: Let T be the set of real numbers, $0 \in$ T such that $|T|>3$ and Γ be the any non empty set. Then T with the ternary operation defined by $x\alpha y\beta z = x$ if $x = y = z$ and $x\alpha y\beta z = 0$ otherwise is a ternary Γ-semigroup.

Example 1.12: T = $\{i, -i\}$ and T = Γ. Then T is a ternary Γ-semigroup under the complex ternary operation (multiplication of complex numbers).

Example 1.13: T = $\{i, 0, -i\}$ and T = Γ. Then T is a ternary Γ-semigroup under the complex ternary operation.

Example 1.14: T = $\{.....,-2i,-i,0,i,2i,.....\}$ and Γ = { *-i, 0, i*}. Then T is a ternary Γ-semigroup under the complex ternary operation.

Example 1.15: T = $\{2x / x \in N\}$ and Γ = N. Then T is a ternary Γ-semigroup under the ternary operation defined by $[a\alpha b\beta c]$ = H.C.F of *a, b* and *c*.

Example 1.16: Let T = $Z^- \times Z^- = \{(a, b) : a, b \in Z^-$, set of all negative integers}and Γ be the any non-empty set. Then T is a ternary semigroup w. r. t the ternary multiplication defined as follows : $(a, b)\alpha(c, d)\beta(e, f) = (a, f)$.

Example 1.17: Let T = {∅, {*a*}, {*b*}, {*c*}, {*a, b*}, {*b, c*}, {*a, c*}, {*a, b, c*}} and Γ = {∅, {*a*}, {*a, b, c*}}. If for all A, C, E ∈ T and B, D ∈ Γ, ABCDE = A ∩ B ∩ C ∩ D ∩ E, then T is a ternary Γ-semigroup.

Example 1.18: Let T = {∅, {*a*}, {*b*}, {*c*}, {*a, b*}, {*b, c*}, {*a, c*}, {*a, b, c*}} and Γ = {{*a, b, c*}}. If for all A, C, E ∈ T and B ∈ Γ, ABCBE = A ∩ B ∩ C ∩ B ∩ E, then T is a ternary Γ-semigroup.

Example 1.19: Let T be the set of all 2 × 3 matrices over Q, the set of rational numbers and Γ be the set of all 3 × 2 matrices over Q. Define AαBβC = usual matrix product of A, α, B, β and C; for all A, B, C ∈ T and for all α, β ∈ Γ. Then T is a ternary Γ-semigroup. Note that T is not a ternary semigroup.

Example: **1.20**: Let T be a ternary Γ-semigroup and α a fixed element in Γ. We define $a.b.c = a\alpha b\alpha c$ for all *a, b, c* ∈ T. We can show that (T, .) is a ternary semigroup and we denote this ternary semigroup by T_α .

Note 1.21: Every ternary semigroup can be considered to be a ternary Γ-semigroup. Thus the class of all ternary Γ-semi groups includes the class of all ternary semi groups.

Example: **1.22 (FREE TERNARY Γ-SEMIGROUP):** Let X and Γ be two nonempty sets. A sequence of elements $a_1\alpha_1a_2\alpha_2 \ldots a_{n-1}\alpha_{n-1}a_n$ where $a_1,a_2,a_3, \ldots\ldots,a_n \in X$ and $\alpha_1,\alpha_2,\alpha_3,\ldots\ldots\ldots.\alpha_n \in \Gamma$ is called a ***word*** over the alphabet X relative to Γ. The set T of all words with the operation defined from T × Γ × T × Γ × T to T as $(a_1\alpha_1a_2\alpha_2 \ldots. a_{n-1}\alpha_{n-1}a_n)\ \gamma(b_1\beta_1b_2\beta_2 \ldots\ldots b_{m-1}\beta_{m-1}b_m)\delta(c_1\varepsilon_1c_2\epsilon_2 \ldots. c_{n-1}\epsilon_{n-1}c_p) = a_1\alpha_1a_2\alpha_2 \ldots\ldots a_{n-1}\alpha_{n-1}a_n\ \gamma b_1\beta_1b_2\beta_2 \ldots\ldots b_{m-1}\beta_{m-1}b_m\ \delta c_1\varepsilon_1c_2\epsilon_2 \ldots. c_{n-1}\epsilon_{n-1}c_p$ is a ternary Γ-semigroup. This ternary Γ-semigroup is called ***free*** **ternary *Γ-semigroup*** over the alphabet X relative to Γ.

In the following we introduce the notion of a commutative ternary Γ-semigroup

Definition 1.23 : A ternary Γ-semigroup T is said to be ***commutative*** provided $a\Gamma b\Gamma c = b\Gamma c\Gamma a = c\Gamma a\Gamma b = b\Gamma a\Gamma c = c\Gamma b\Gamma a = a\Gamma c\Gamma b$ for all $a,b,c \in$ T.

Example 1.24 : T = $\{0, \pm i\}$ and Γ = { 0, *i* } with complex ternary operation is a commutative ternary Γ-semigroup.

Note 1.25 : Let T be a ternary Γ-semigroup and *a, b* ∈ T and α ∈ Γ. Then $a\alpha a\alpha b$ is denoted by $(a\alpha)^2b$ and consequently $a\ \alpha\ a\ \alpha\ a\ \alpha \ldots..(n\ terms)b$ is denoted by $(a\alpha)^nb$.

In the following we introduce a quasi commutative ternary Γ-semigroup.

Definition 1.26 : A ternary Γ-semigroup T is said to be ***quasi commutative*** provided for each *a, b, c* ∈ T and $\gamma \in \Gamma$, there exists a natural number *n* such that $a\gamma b\gamma c = (b\gamma)^n a\gamma c = b\gamma c\gamma a = (c\gamma)^n b\gamma a = c\gamma a\gamma b = (a\gamma)^n c\gamma b$.

Note 1.27 : If a ternary Γ-semigroup T is ***quasi commutative*** then for each *a,b* ∈ T , there exists a natural number *n* such that, $a\Gamma b\Gamma c = (b\Gamma)^n a\Gamma c = b\Gamma c\Gamma a = (c\Gamma)^n b\Gamma a = c\Gamma a\Gamma b = (a\Gamma)^n c\Gamma b$.

Theorem 1.28: If T is a commutative ternary Γ-semigroup then T is a quasi commutative ternary Γ-semigroup.

Proof : Suppose that T is a commutative ternary Γ-semigroup. Let $a, b, c \in$ T and $\gamma \in \Gamma$.

Now $a\gamma b\gamma c = b\gamma c\gamma a = c\gamma a\gamma b \Rightarrow a\gamma b\gamma c = (b\gamma)^1 a\gamma c = b\gamma c\gamma a = (c\gamma)^1 b\gamma a = c\gamma a\gamma b = (a\gamma)^1 c\gamma b$. T is a quasi commutative ternary Γ-semigroup.

In the following we introduce the notion of a normal ternary Γ-semigroup.

Definition 1.29 : A ternary Γ-semigroup T is said to be ***normal*** provided $a\alpha b\beta T = T\beta a\alpha b\ \forall\ a,b \in$ T and $\alpha, \beta \in \Gamma$.

Note 1.30 : If a ternary Γ-semigroup T is *normal* then $a\Gamma b\Gamma T = T\Gamma a\Gamma b\ \forall\ a, b \in$ T.

Theorem 1.31 : If T is a quasi commutative ternary Γ-semigroup then T is a normal ternary Γ-semigroup.

***Proof* :** $a, b \in$ T and $\alpha, \beta \in \Gamma$. If $x \in a\alpha b\beta T$. Then $x = a\alpha b\beta c$ where *c* ∈ T. Since T is quasi commutative ternary Γ-semigroup $a\alpha b\beta c = (c\alpha)^n a\beta b = (c\alpha)^{n-1}c\alpha a\beta b \in T\Gamma a\Gamma b$. Therefore $x \in T\Gamma a\Gamma b$. Thus $a\Gamma b\Gamma T \subseteq T\Gamma a\Gamma b$. Similarly $T\Gamma a\Gamma b \subseteq a\Gamma b\Gamma T$ and hence $a\Gamma b\Gamma T = T\Gamma a\Gamma b\ \forall\ a,b \in$ T.

Corollary 1.32 : Every commutative ternary Γ-semigroup is a normal ternary Γ-semigroup.

In the following we are introducing left pseudo commutative ternary Γ-semigroup.

Definition 1.33 : A ternary Γ-semigroup T is said to be ***left pseudo commutative*** provided $a\Gamma b\Gamma c\Gamma d\Gamma e = b\Gamma c\Gamma a\Gamma d\Gamma e = c\Gamma a\Gamma b\Gamma d\Gamma e = b\Gamma a\Gamma c\Gamma d\Gamma e = c\Gamma b\Gamma a\Gamma d\Gamma e = a\Gamma c\Gamma b\Gamma d\Gamma e\ \forall\ a,b,c,d,e \in$ T.

Theorem 1.34 : If T is a commutative ternary semigroup, then T is a left pseudo commutative ternary Γ-semigroup.

Proof: Suppose that T is a commutative ternary semigroup. Then $a\Gamma b\Gamma c\Gamma d\Gamma e = (a\Gamma b\Gamma c)\Gamma d\Gamma e = (b\Gamma c\Gamma a)\Gamma d\Gamma e = (c\Gamma a\Gamma b)\Gamma d\Gamma e = (b\Gamma a\Gamma c)\Gamma d\Gamma e = (c\Gamma b\Gamma a)\Gamma d\Gamma e = (a\Gamma c\Gamma b)\Gamma d\Gamma e\ \forall\ a,b,c,d,e \in$ T. $a\Gamma b\Gamma c\Gamma d\Gamma e = b\Gamma c\Gamma a\Gamma d\Gamma e = c\Gamma a\Gamma b\Gamma d\Gamma e = b\Gamma a\Gamma c\Gamma d\Gamma e = c\Gamma b\Gamma a\Gamma d\Gamma e = a\Gamma c\Gamma b\Gamma d\Gamma e$. T is a left pseudo commutative ternary Γ-semigroup.

Note 1.35 : The converse of the above theorem is not true. i.e T is a left pseudo commutative ternary Γ-semigroup then T need not be a commutative ternary Γ-semigroup.

Example 1.36 : Let T = $\{a,b,c,d,e\}$ and Γ = {α}. Define a ternary operation [ ] on T as $[abc]$ = *a.b.c* where the binary operation 'α' is defined as

α	a	b	c	d	e
a	a	a	a	a	a
b	b	a	a	a	a
c	a	a	a	a	a
d	a	a	a	a	a
e	a	b	c	d	e

It is easy to see that T is a ternary Γ-semigroup. Now T is a left pseudo commutative ternary Γ-semigroup. But T is not a commutative ternary Γ-semigroup.

In the following we are introducing the notion of lateral pseudo commutative ternary Γ-semigroup.

Definition 1.37 : A ternary Γ-semigroup T is said to be a ***lateral pseudo commutative*** ternary Γ-semigroup provide $a\Gamma b\Gamma c\Gamma d\Gamma e$ = $a\Gamma c\Gamma d\Gamma b\Gamma e$ = $a\Gamma d\Gamma b\Gamma c\Gamma e$ = $a\Gamma c\Gamma b\Gamma d\Gamma e$ = $a\Gamma d\Gamma c\Gamma b\Gamma e$ = $a\Gamma b\Gamma d\Gamma c\Gamma e$ for all $a,b,c,d,e \in$ T.

Theorem 1.38 : If T is a commutative ternary semigroup then T is a lateral pseudo commutative ternary Γ-semigroup.

Proof : Suppose that T is a commutative ternary Γ-semigroup. Then $a\Gamma b\Gamma c\Gamma d\Gamma e$ = $a\Gamma(b\Gamma c\Gamma d)\Gamma e$ = $a\Gamma(c\Gamma d\Gamma b)\Gamma e$ = $a\Gamma(d\Gamma b\Gamma c)\Gamma e$ = $a\Gamma(c\Gamma b\Gamma d)\Gamma e$ = $a\Gamma(d\Gamma c\Gamma b)\Gamma e$ = $a\Gamma(b\Gamma d\Gamma c)\Gamma e$ for all $a,b,c,d,e \in$ T.

$a\Gamma b\Gamma c\Gamma d\Gamma e$ = $a\Gamma c\Gamma d\Gamma b\Gamma e$ = $a\Gamma d\Gamma b\Gamma c\Gamma e$ = $a\Gamma c\Gamma b\Gamma d\Gamma e$ = $a\Gamma d\Gamma c\Gamma b\Gamma e$ = $a\Gamma b\Gamma d\Gamma c\Gamma e$. Therefore T is a lateral pseudo commutative ternary Γ-semigroup.

Note 1.39 : The converse of the above theorem is not true i.e. T is a lateral pseudo commutative ternary Γ-semigroup then T need not be a commutative ternary Γ-semigroup.

Example 1.40 : Consider the ternary Γ-semigroup in example 1.36, T is a lateral pseudo commutative. But T is not a commutative ternary Γ-semigroup.

In the following we are introducing the notion of right pseudo commutative ternary Γ-semigroup.

Definition 1.41 : A ternary Γ-semigroup T is said to be ***right pseudo commutative*** provided $a\Gamma b\Gamma c\Gamma d\Gamma e = a\Gamma b\Gamma d\Gamma e\Gamma c = a\Gamma b\Gamma e\Gamma c\Gamma d = a\Gamma b\Gamma d\Gamma c\Gamma e = a\Gamma b\Gamma e\Gamma d\Gamma c = a\Gamma b\Gamma c\Gamma e\Gamma d$ $\forall$ $a, b, c, d, e \in$ T.

Theorem 1.42 : If T is a commutative ternary Γ-semigroup then T is a right pseudo commutative ternary Γ-semigroup.

Proof : Suppose that T is a commutative ternary semigroup. Then $a\Gamma b\Gamma c\Gamma d\Gamma e$ = $a\Gamma b\Gamma(c\Gamma d\Gamma e)$ = $a\Gamma b\Gamma(d\Gamma e\Gamma c)$ = $a\Gamma b\Gamma(e\Gamma c\Gamma d)$ = $a\Gamma b\Gamma(d\Gamma c\Gamma e)$ = $a\Gamma b\Gamma(e\Gamma d\Gamma c)$ = $a\Gamma b\Gamma(c\Gamma e\Gamma d)$ for all $a,b,c,d,e \in$ T.

$a\Gamma b\Gamma c\Gamma d\Gamma e$ = $a\Gamma b\Gamma d\Gamma e\Gamma c$ = $a\Gamma b\Gamma e\Gamma c\Gamma d$ = $a\Gamma b\Gamma d\Gamma c\Gamma e$ = $a\Gamma b\Gamma e\Gamma d\Gamma c$ = $a\Gamma b\Gamma c\Gamma e\Gamma d$. T is a right pseudo commutative ternary Γ-semigroup.

Note 1.43 : The converse of the above theorem is not true i.e. If T is a right pseudo commutative ternary Γ-semigroup, then T need not be a commutative ternary Γ-semigroup.

Example 1.44 : Consider the ternary Γ-semigroup in example 1.36, T is a right pseudo commutative. But T is not a commutative ternary Γ-semigroup.

In the following we introducing the notion of pseudo commutative ternary Γ-semigroup.

Definition 1.45: A ternary Γ-semigroup T is said to be ***pseudo commutative***, provided T is a left pseudo commutative, right pseudo commutative and lateral pseudo commutative ternary Γ-semigroup.

Theorem 1.46: If T is a commutative ternary Γ-semigroup, then T is a pseudo commutative ternary Γ-semigroup.

Proof: Suppose that T is a commutative ternary Γ-semigroup. By theorem 1.34, T is a left pseudo commutative ternary Γ-semigroup. By theorem 1.42, T is a right pseudo commutative ternary Γ-semigroup. By theorem 1.38, T is a lateral pseudo commutative ternary Γ-semigroup. T is a pseudo commutative ternary Γ-semigroup.

Note 1.47 : The converse of the above theorem is not true i.e. if T is a pseudo commutative ternary Γ-semigroup , then T need not be a commutative ternary Γ-semigroup.

ExamplE 1.48 : Consider the ternary Γ-semigroup in example 1.36, T is a pseudo commutative. But T is not a commutative ternary Γ-semigroup.

2. TERNARY Γ-SUB SEMIGROUP

In the following we are introducing a ternary Γ-sub semigroup

Definition 2.1: Let T be ternary Γ-semigroup. A non empty subset 'S' is said to be a ***ternary Γ-subsemigroup*** of T if $a\alpha b\beta c$ $\in$ S for all $a, b, c \in$ S and $\alpha, \beta \in \Gamma$.

Note 2.2: A non empty subset S of a ternary Γ-semigroup T is a ternary Γ-subsemigroup if and only if SΓSΓS $\subseteq$ S.

Example 2.3 : Let S = [0,1] and Γ = { 1/*n* : *n* is a positive integer}. Then S is a ternary Γ-semigroup under the usual multiplication. Let T = [0, 1/2]. Now T is a nonempty subset of S and $a\gamma b\delta c \in$ T, for all $a, b \in$ T and γ , $\delta \in \Gamma$. Then T is a ternary **Γ**-subsemigroup of S.

Example 2.4 : Let T = { *a*, *b*, *c*, *d* } and Γ = { α } be a Γ-semigroup under the operation **.** given by

α	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	a

Define the ternary operation [ ] as $[x\alpha y\alpha z] = x(y\alpha z) = (x\alpha y)\alpha z$. Then S is a ternary Γ-semigroup. Let A = {*a* }, B = {*a*, *b*}, C = {*a*, *b*, *c*} and D = {*a*, *b*, *d* }. Then A, B, C, D are all ternary Γ-subsemigroups of T.

Example 2.5 : Consider the ternary Γ-semigroup Z^- under the ternary multiplication, then S = Z^- \{-1} is ternary Γ-subsemigroup of Z^-.

Theorem 2.6: The non-empty intersection of two ternary Γ-subsemigroups of a ternary Γ-semigroup T is a ternary Γ-subsemigroup of T.

Proof : Let S_1, S_2 be two ternary Γ-subsemigroups of T . Let *a*, *b*, *c*$\in S_1 \cap S_2$ and $\alpha, \gamma \in \Gamma$.

$a, b, c \in S_1 \cap S_2 \Rightarrow a, b, c \in S_1$ and $a, b, c \in S_2$

$a,b,c \in S_1$ and $\alpha, \gamma \in \Gamma$, S_1 is a ternary Γ-subsemigroup of T

$\Rightarrow a\alpha b\gamma c \in S_1$

$a, b, c \in S_2$ $\alpha, \gamma \in \Gamma$, S_2 is a ternary Γ-subsemigroup of T $\Rightarrow a\alpha b\gamma c \in S_2$

$a\alpha b\gamma c \in S_1$, $a\alpha b\gamma c \in S_2 \Rightarrow a\alpha b\gamma c \in S_1 \cap S_2$.

Therefore $S_1 \cap S_2$ is a ternary Γ-subsemigroup of T.

Theorem 2.7: The non-empty intersection of any family of ternary Γ-subsemigroups of a ternary Γ-semigroup T is a ternary Γ-subsemigroup of T.

***Proof*:** Straight forward.

In the following we are introducing a ternary Γ-subsemigroup which is generated by a subset and a cyclic ternary Γ-subsemigroup of ternary Γ-semigroup.

Definition 2.8: Let T be a ternary Γ-semigroup and A be a non-empty subset of T. The smallest ternary Γ-subsemigroup of T containing A is called a ***ternary Γ-subsemigroup of T generated by A***. It is denoted by (A).

Theorem 2.9: Let T be a ternary Γ-semigroup and A be a non-empty subset of T. Then (A) = { $a_1\ \alpha_1\ a_2\ \alpha_2 \ldots\ldots a_{n-1}\ \alpha_{n-1}\ a_n$: for some odd natural number *n*, $a_1, a_2, \ldots, a_n \in A$, $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \Gamma$ }.

***Proof* :** Let S = { $a_1 \alpha_1 a_2\ \alpha_2 \ldots a_{n-1}\ \alpha_{n-1}\ a_n$: $n \in N$, $a_1, a_2, \ldots a_n \in A$, $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \Gamma$}. Let $a, b, c \in S$ and $\alpha, \gamma \in \Gamma$.

$a \in T \Rightarrow a = a_1\ \alpha_1\ a_2\ \alpha_2 \ldots\ldots a_{m-1}\ \alpha_{m-1}\ a_m$ where $a_1, a_2, \ldots a_m \in A$, $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma$.

$b \in T \Rightarrow b = b_1\ \beta_1\ b_2\ \beta_2 \ldots\ldots b_{n-1}\ \beta_{n-1}\ b_n$ where $b_1, b_2, \ldots b_n \in A$, $\beta_1, \beta_2, \ldots, \beta_{n-1} \in \Gamma$.

$c \in S \Rightarrow c = c_1\gamma_1 c_2\gamma_2 c_3 \ldots \gamma_{r-1} c_r$ where $c_1, c_2 \ldots c_r \in A$, $\gamma_1, \gamma_2, \ldots, \gamma_{n-1} \in \Gamma$.

Now $a\gamma b\alpha c$ = $(a_1\alpha_1 a_2\alpha_2 \ldots a_{m-1}\alpha_{m-1}a_m)$ γ $(b_1\beta_1 b_2\beta_2 \ldots b_{n-1}\beta_{n-1}b_n)$($c_1\gamma_1 c_2\gamma_2 c_3 \ldots \gamma_{r-1} c_r$) ∈ S.

Therefore S is a ternary Γ-subsemigroup of T.

Let K be a ternry Γ-subsemigroup of S such that A ⊆ K.

Let $a \in S$. Then $a = a_1\ \alpha_1\ a_2\ \alpha_2 \ldots\ldots a_{n-1}\ \alpha_{n-1}\ a_n$ where $a_1, a_2, \ldots a_n \in A$, $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \Gamma$

$a_1, a_2, \ldots a_n \in A$, $A \subseteq K \Rightarrow a_1, a_2, \ldots a_n \in K$.

$a_1, a_2, \ldots a_n \in K$, $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \Gamma$, K is a ternary Γ-subsemigroup

$\Rightarrow a_1\ \alpha_1\ a_2\ \alpha_2 \ldots\ldots a_{n-1}\ \alpha_{n-1}\ a_n \in K \Rightarrow a \in K$. Therefore S⊆ K.

So S is the smallest ternary Γ-subsemigroup of T containing A. Hence (A) = S.

Theorem 2.10 : Let T be a ternary Γ-semigroup and A be a non-empty subset of T. Then (A) = the intersection of all ternary Γ-subsemigroups of T containing A.

Proof : Let Δ be the set of all ternary Γ-subsemigroups of T containing A. Since T is a ternary Γ-subsemigroup of T containing A , T∈ Δ, so $\Delta \neq \varnothing$

Let $S^* = \bigcap_{S \in \Delta} S$. Since A ⊆S for all S∈ Δ and A ⊆ S^*

By theorem 2.7 , S^* is a ternary Γ-subsemigroup of T.

Since S^* ⊆ S for all S∈ Δ , S^* is the smallest ternary Γ-subsemigroup of T containing A. There fore S^* = (A).

Definition 2.11 : Let T be a ternary Γ-semigroup. A ternary Γ-subsemigroup S of T is said to be a ***cyclic ternary Γ-subsemigroup*** of T if S is generated by a single element subset of T.

Definition 2.12 : A ternary semigroup T is said to be a ***cyclic ternary semigroup*** if T is cyclic ternary subsemigroup of T itself.

3.<u>SPECIAL ELEMENTS OF A TERNARY Γ-SEMIGEOUP</u>

In the following we introducing left identity, lateral identity, right identity, two sided identity and identity of ternary Γ-semigroup.

Definition 3.1 : An element *a* of ternary Γ-semigroup T is said to be ***left identity*** of T provided $a\alpha a\beta t = t$ for all $t \in T$, $\alpha, \beta \in \Gamma$.

Note 1.3.2 : Left identity element *a* of a ternary Γ-semigroup T is also called as ***left unital element.***

Definition 3.3 : An element *a* of a ternary Γ-semigroup T is said to be a ***lateral identity*** of T provided $a\alpha t\beta a = t$ for all $t \in T$, $\alpha, \beta \in \Gamma$.

Note 3.4 : Lateral identity element *a* of a ternary Γ-semigroup T is also called as ***lateral unital element***.

Definition 3.5: An element *a* of a ternary Γ-semigroup T is said to be a ***right identity*** of T provided $t\alpha a\beta a = t\ \forall\ t \in T$, $\alpha, \beta \in \Gamma$.

Note 3.6 : Right identity element *a* of a ternary Γ-semigroup T is also called as ***right unital element.***

Definition 3.7 : An element *a* of a ternary Γ-semigroup T is said to be a ***two sided identity*** of T provided $a\alpha a\beta t = \alpha t a\beta a = t\ \forall\ t \in T$, $\alpha, \beta \in \Gamma$.

Note 3.8 : Two-sided identity element of a ternary Γ-semigroup T is also called as ***bi-unital element***.

Definition 3.9 : An element *a* of a ternary Γ-semigroup T is said to be an ***identity*** provided $a\alpha a\beta t = t\alpha a\beta a = a\alpha t\beta a = t\ \forall\ t \in T$, $\alpha, \beta \in \Gamma$.

Note 3.10 : An identity element of a ternary Γ-semigroup T is also called as ***unital element.***

Note 3.11 : An element *a* of a ternary Γ-semigroup T is an ***identity*** of T iff *a* is left identity , lateral identity and right identity of T.

Example 3.12 : Let Z_0^- be the set of all non-positive integers and Γ be the set of binary operations. Then with the usual ternary multiplication, Z_0^- forms a ternary Γ-semigroup with identity element -1.

Note 3.13 : The identity (if exists) of a ternary Γ-semigroup is usually denoted by 1 (or) e.

Definition 3.14 : A ternary Γ-semigroup T with identity is called a ***ternary Γ-monoid***.

Notation 3.15 : Let T be a ternary Γ-semigroup. If T has an identity, let T^1 = T and if T does not have an identity , let T^1 be the ternary semigroup T with an identity adjoined usually denoted by the symbol 1.

In the following we introducing left zero, lateral zero, right zero, two sided zero and zero of ternary Γ-semigroup.

Definition 3.16 : An element *a* of a ternary Γ-semigroup T is said to be a ***left zero*** of T provided $a\alpha b\beta c = a\ \forall\ b, c \in T$, $\alpha, \beta \in \Gamma$.

Definition 3.17 : An element *a* of a ternary Γ-semigroup T is said to be a ***lateral zero*** of T provided $b\alpha a\beta c = a\ \forall\ b, c \in T$, $\alpha, \beta \in \Gamma$.

Definition 3.18 : An element a of a ternary Γ-semigroup T is said to be a ***right zero*** of T provided $b\alpha c\beta a = a \ \forall \ b, c \in$ T, $\alpha, \beta \in \Gamma$.

Definition 3.19 : An element a of a ternary Γ-semigroup T is said to be a ***two sided zero*** of T provided $a\alpha b\beta c = b\alpha c\beta a = a \ \forall$ $b, c \in$ T, $\alpha, \beta \in \Gamma$.

Note 3.20 : If a is a two sided zero of a ternary Γ-semigroup T, then a is both left zero and right zero of T.

Definition 3.21 : An element a of a ternary Γ-semigroup T is said to be ***zero*** of T provided $a\alpha b\beta c = b\alpha a\beta c = b\alpha c\beta a = a \ \forall \ b,$ $c \in$ T, $\alpha, \beta \in \Gamma$.

Note 3.22 : If a is a zero of T, then a is a left zero, lateral zero and right zero of T.

Theorem 3.23 : If a is a left zero, b is a lateral zero and c is a right zero of a ternary Γ-semigroup T , then $a = b = c$.

Proof : Since a is a left zero of T, $a\alpha b\beta c = a$ for all $a,b, c \in$ T. Since b is a lateral zero of T, $a\alpha b\beta c = b$. Since c is a right zero of T, $a\alpha b\beta c = c$. Therefore $a\alpha b\beta c = a = b = c$.

Theorem 3.24 : Any ternary Γ-semigroup has at most one zero element.

Proof : Let a, b, c be three zeros of a ternary Γ-semigroup T. Now a can be considered as a left zero , b can be considered as a lateral zero and c can be considered as a right zero of T. By theorem 3.24, $a = b = c$. Then T has at most one zero.

Note 3.25 : The zero (if exists) of a ternary Γ-semigroup is usually denoted by 0.

Notation 3.26 : Let T be a ternary Γ-semigroup. if T has a zero, let T^0 = T and if T does not have a zero , let T^0 be the ternary Γ-semigroup T with zero adjoined usually denoted by the symbol 0.

In the following we introducing the notion of left zero ternary Γ-semigroup, lateral zero ternary Γ-semigroup, right zero ternary Γ-semigroup and zero ternary Γ-semigroup.

Definition 327 : A ternary Γ-semigroup in which every element is a left zero is called a ***left zero ternary Γ-semigroup***.

Definition 3.28 : A ternary Γ-semigroup in which every element is a lateral zero is called a ***lateral zero ternary Γ-semigroup.***

Definition 3.29 : A ternary Γ-semigroup in which every element is a right zero is called a ***right zero ternary Γ-semigroup***.

Definition 3.30 : A ternary Γ-semigroup with 0 in which the product of any three elements equal to 0 is called ***a zero ternary Γ-semigroup*** (or) **null ternary Γ-semigroup**.

Example 3.31 : Let $0 \in$ T ⊆ R and $|T| > 2$ and Γ be the any non-empty set. Then T with the ternary operation defined by $x\alpha y\beta z = x$ if $x = y = z$ and $x\alpha y\beta z = 0$ otherwise is a ternary Γ-semigroup with zero.

In the following we are introducing the notion of idempotent element of a ternary Γ-semigroup.

Definition 3.32 : An element a of a ternary Γ-semigroup T is said to be an ***α-idempotent*** element provided $a\alpha a\alpha a = a$.

Note 3.33 : The set of all idempotent elements in a ternary Γ-semigroup T is denoted by E_α(T).

Example 3.34 : Every identity, zero elements are α-idempotent elements.

Definition 3.35 : An element a of a ternary Γ-semigroup T is said to be an **(α, β)-*idempotent*** element provide $a\alpha a\beta a = a$ for all $\alpha, \beta \in \Gamma$.

Note 3.36 : In a ternary Γ-semigroup T, ***a*** is an idempotent iff a is an (α, β)-idempotent for all $\alpha, \beta \in \Gamma$.

Note 3.37 : If an element ***a*** of a ternary Γ- semigroup T is *an* ***idempotent,*** then $a\Gamma a\Gamma a = a$.

In the following we introduce proper idempotent element ternary Γ-semigroup.

Definition 3.38 : An element a of a ternary Γ-semigroup T is said to be a ***proper idempotent*** element provided a is an idempotent which is not the identity of T if identity exists.

We now introduce an idempotent ternary Γ-semigroup and a strongly idempotent ternary Γ-semigroup.

Definition 3.39 : A ternary Γ-semigroup T is said to be an ***idempotent ternary Γ-semigroup*** provided every element of S is α –idempotent for some $\alpha \in \Gamma$.

Definition 3.40 : A ternary Γ-semigroup T is said to be a ***strongly idempotent ternary Γ- semigroup*** or ***ternary Γ-band*** provided every element in T is an α –idempotent for some $\alpha \in \Gamma$.

In the following we are introducing regular element and regular ternary Γ-semigroup.

Definition 3.41 : An element a of a ternary Γ-semigroup T is said to be ***regular*** if there exist $x, y \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a\alpha x\beta a\gamma y\delta a = a$.

Definition 3.42 : A ternary Γ-semigroup T is said to be ***regular ternary Γ-semigroup*** provided every element is regular.

Example 3.43: Let T = {0, a, b} and Γ be any nonempty set. If we define a binary operation on T as the following Cayley table, then T is a ternary Γ-semigroup.

.	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Define a mapping from T × Γ × T × Γ × T to T as $a\alpha b\beta c = abc$ for all $a, b, c \in$ T and $\alpha, \beta \in \Gamma$. Then T is regular ternary Γ-semigroup.

Theorem 3.44 : Every α-idempotent element in a ternary Γ-semigroup is regular.

Proof : Let a be an α-idempotent element in a ternary Γ-semigroup T. Then $a = a\alpha a\alpha a = (a\alpha a\alpha a)a\alpha a = a\alpha a\alpha a\alpha a\alpha a$. Therefore a is regular element.

In the following we are introducing the notion of left regular, lateral regular right regular, intra regular and completely regular elements of a ternary Γ-semigroup and completely regular ternary Γ-semigroup.

Definition 3.45 : An element a of a ternary Γ-semigroup T is said to be ***left regular*** if there exist $x, y \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = a\alpha a\beta a\gamma x\delta y$. i.e, $a \in a\Gamma a\Gamma a\Gamma T\Gamma T$.

Definition 3.46 : An element a of a ternary Γ-semigroup T is said to be ***lateral regular*** if there exist $x, y \in$ T and $\alpha, \beta, \gamma, \delta \in$ Γ such that $a = x\alpha a\beta a\gamma a\delta y$. i.e., $a \in T\Gamma a\Gamma a\Gamma a\Gamma T$.

Definition 3.47 : An element a of a ternary Γ-semigroup T is said to be ***right regular*** if there exist $x, y \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = x\alpha y\beta a\gamma a\delta a$. i.e., $a \in T\Gamma T\Gamma a\Gamma a\Gamma a$.

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Definition 3.48 : An element a of a ternary Γ-semigroup T is said to be ***completely regular*** if there exist $x, y \in T$ $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a\alpha x\beta a\gamma y\delta a = a$ and $a\alpha x\beta a = x\alpha a\beta a = a\alpha a\beta x = a\alpha y\beta a = y\alpha a\beta a = a\alpha a\beta y = a\alpha x\beta y = y\alpha x\beta a = x\alpha a\beta y = y\alpha a\beta x$.

Note 3.49: An element a of a ternary Γ-semigroup T is said to be ***completely regular*** if there exist $x, y \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a \in a\Gamma x\Gamma a\Gamma y\Gamma a$ and $a\Gamma x\Gamma a = x\Gamma a\Gamma a = a\Gamma a\Gamma x = a\Gamma y\Gamma a = y\Gamma a\Gamma a = a\Gamma a\Gamma y = a\Gamma x\Gamma y = y\Gamma x\Gamma a = x\Gamma a\Gamma y = y\Gamma a\Gamma x$.

Definition 3.50 : A ternary Γ-semigroup T is said to be a ***completely regular ternary Γ-semigroup*** provided every element in T is completely regular.

Definition 3.51 : An element a of a ternary Γ-semigroup T is said to be ***intra regular*** if there exist $x, y \in T$ and $\alpha, \beta, \gamma, \delta, \varepsilon, \epsilon \in \Gamma$ such that $a = x\alpha a\beta a\gamma a\delta a\varepsilon a\epsilon y$.

Theorem 3.52 : Let T be a ternary Γ-semigroup and $a \in$ T. If a is a completely regular element, then a is regular, left regular, lateral regular and right regular.

Proof **:** Suppose that a is completely regular.

Then there exist $x, y \in T$ and for all $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a\alpha x\beta a\gamma y\delta a = a$ and $a\alpha x\beta a = x\alpha a\beta a = a\alpha a\beta x = a\alpha y\beta a = y\alpha a\beta a = a\alpha a\beta y = a\alpha x\beta y = y\alpha x\beta a = x\alpha a\beta y = y\alpha a\beta x$. Clearly A is regular.

Now $a = a\alpha x\beta a\gamma y\delta a = a\alpha x\beta a\gamma a\delta y = a\alpha a\beta a\gamma x\delta y$. Therefore a is left regular.

Also $a = a\alpha x\beta a\gamma y\delta a = x\alpha a\beta a\gamma y\delta a = x\alpha a\beta a\gamma a\delta y$. Therefore a is lateral regular.

and $a = a\alpha x\beta a\gamma y\delta a = x\alpha a\beta a\gamma y\delta a = x\alpha y\beta a\gamma a\delta a$. Therefore a is right regular.

In the following we are introducing the notion of mid unit of a ternary Γ-semigroup.

Definition 3.53 : An element a of a ternary Γ-semigroup T is said to be a ***mid-unit*** provided $x\Gamma a\Gamma y\Gamma a\Gamma z = x\Gamma y\Gamma z$ for all $x, y, z \in T$.

CONCLUSION

D. Madhusudhana Rao and A. Anjaneyulu studied about Γ-semigroups. Further D. Madhusudhana Rao and A .Anjaneyulu and Y. Sarla extended the same results to ternary semigroups. In this paper mainly we extended the same results to ternary Γ-semigroups.

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