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Development and implementation of BEM solution for 3- D Arbitrary shaped conduction bodies using face based basis functions

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Abstract--In this work, due to rapid development of BEM , thermal conduction problem is solved using the boundary integral equation formulation and boundary element solution method. Arbitrary bodies which are three dimensional in shape are analyzed and addressed. The geometry of the body is divided into triangular patches using triangular patch modeling. A numerical solution is developed by defining the basis functions on the face of the triangles generated in the triangular patch modeling, in contrast to defining it on the nodes as followed in the regular BEM solutions. The temperature distribution from the surface of the hot body is plotted for different boundary temperature. Also the convergence study is conducted to get the solution convergence towards the exact solution for the case of a sphere. Other geometries bodies like cube and cylinder are also treated.

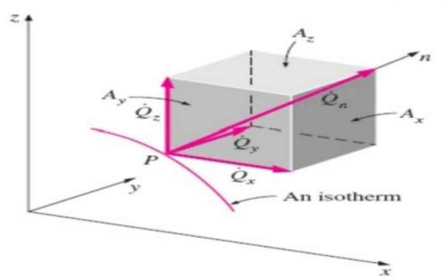
Keywords--Boundary Integral Equations, Thermal Conduction, Face Based Basis Functions, Error Analysis

I. Introduction

Heat conduction is one of three modes of heat transfer and plays a crucial role in the application like heat sinks, heat engines, electronic cooling etc. The heat conduction is part of almost all the heat transfer processes and it is difficult to find a heat transfer application without the presence of heat conduction. Heat conduction is governed by the Fourier law of heat conduction. As per the law, heat transfer into the body or out of the body through heat conduction process depends on the geometry of the body, thermal conductivity and temperature gradient.

The analysis of basic mechanism of heat transfer in solids that is heat conduction problem, insignificant for process of designing and optimization mechanical systems and devices. Accordingly, the heat conduction equations with conditions of variable temperature or heat flux on boundaries become an important instrument for mathematical description of many engineering, geothermal and biological problems. As a result, there is a need to develop effective computational methods and tools for solving transient heat conduction problem. Conduction of heat means transfer of heat energy within the body due to the temperature gradient. Heat spontaneously flows from a body having higher temperature to lower temperature. But in absence of external driving fluxes it approaches to thermal equilibrium.

There are two types of conduction such as:
1) Steady state thermal conduction and
2) Transient or Unsteady state conduction



Steady state conduction:

Steady state conduction is a form of conduction where the temperature differences deriving by the conduction remains constant and it is independent of time. Steady state conduction happens when the temperature difference driving the conduction are constant after an equilibration time, the spatial distribution of temperatures (i.e. temperature field ΔT)in the conducting object does not change further. Thus, all partial derivatives of temperature with respect to space may either be zero or have nonzero values, but all derivatives of temperature at any point with respect to time are uniformly zero. Instead state conduction, the amount of heat entering any region of an object is equal to amount of heat coming out. The steady state heat conduction problem is well known. The analysis will be difficult for the problems like, where heat transfer takes place through a complicated domain. Different numerical methods are proposed for these types of problems, e.g. finite difference method, finite volume method and finite element method.The heat transfer phenomenon through conduction is usually represented in the differential equation form along with the boundary conditions. The boundary conditions can

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be either essential boundary conditions or natural boundary conditions or mixed boundary conditions. In essential boundary conditions, temperature is specified on the boundary; whereas in natural boundary conditions, heat flux is specified. In case of mixed boundary conditions, temperature is specified on a part of the boundary and heat flux is specified on remaining part of the boundary.

In case on numerical solution procedure, the geometry of the body is divided into the known shapes. The process of dividing the geometry into known shapes (also known as elements) is referred to as the discretization of the geometry. The governing equations are solved on these known geometrical shapes and the temperatures are determined. When the governing equations are expressed in the form of differential equations, then solution domain is the volume of the geometry in which the temperature distribution is to be found. Several numerical methods that use the method of solving the differential equation of the heat conduction are Finite Element Method (FEM), Finite Difference Method (FDM), Finite Volume Methods (FVM) and Spectral Methods(SM) etc. The main drawback in using the differential equation based numerical solution methods is, if the solution domain is a open domain, then the number of basic elements that are required to be generated is enormously high for an accurate solution, which makes the computational time very intensive. Example of these kind of applications include heat transfer in geometries having very intricate shapes by closed domain problems, open domain problems like thermal distribution in the sea when ship is moving around so that one can estimate the thermal effects on the marine life etc.

To overcome this problem of high computational cost when dealing with the open domain problems, one can use the integral equation formulation instead of differential equation formulation. In integral equation formulation, only the boundary of the geometry is discretized if it is a open domain problem or a closed domain problem and hence number elements generated in the boundary is limited. The numerical solution method for the boundary integral equation methods is known as boundary element method, boundary integral method or method of moments solution procedure.

Boundary element method is popular in dealing with the structural analysis and is a well known numerical method in that area [1,2]. In the BEM solution proposed in [1,2], the basis function are defined with respect to the nodes similar to the FEM. Authors in ref [3,4] introduced a novel method of defining the basis function on the triangular face of the elements in contrast to defining it with respect to the nodes. In the solution methodology proposed by authors [3,4], there is no concept of nodes required to define the basis functions. Later the basis functions were defined on the edges [5] and then on the nodes [6]. But the basis functions defined in the research work [6] is quite different from what is used in ref [1,2]. However the methods proposed in ref [3-6] are applied to the acoustic scattering problems rather than heat conduction problem. In this research work it is proposed to the face based basis functions to solve the heat conduction problem. The basis functions defined in this work are similar to the one proposed in ref [3] but for a heat conduction problem rather than for an acoustic problem. Other numerical methods that are available [5-15] to address the problem of heat conduction.

II. General Procedure for BEM

For an equation resulting from governing equations, which is in the form of

$$\boldsymbol{L}\,\boldsymbol{f} = \boldsymbol{g} \qquad (1)$$

where

$\boldsymbol{L}$: Linear operator,

$\boldsymbol{g}$: Known function resulting from the forcing function

$\boldsymbol{f}$:Unknown function to be determined,

The solution can be derived as follows:

Let the unknown function $\boldsymbol{f}$ be approximated by a set of known functions $\boldsymbol{f}_j$, $j = 1,2,\ldots,N$

$$\boldsymbol{f} = \sum_{n=1}^{N} \beta_j \boldsymbol{f}_j \qquad (2)$$

where

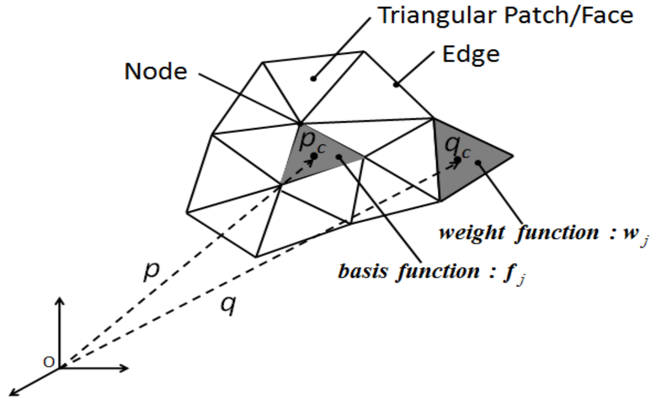
$\boldsymbol{f}_j$: Basis functions in the domain of $\boldsymbol{L}$

β_j are scalar coefficients to be determined.

Substituting Eq. 2 into Eq. 1

$$\sum_{n=1}^{N} \beta_j \boldsymbol{L}\,\boldsymbol{f}_j = \boldsymbol{g} \qquad (3)$$

where the equality is usually approximate.



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Let $\boldsymbol{w}_i$ testing functions in the range of $\boldsymbol{L}$. Now, taking the inner product of Eq. 3 with each w_i and using the linearity of inner product defined as $\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_s \boldsymbol{f} \bullet \boldsymbol{g} \; ds$, we obtain a set of linear equations, given by

$$\sum_{n=1}^{N} \beta_j \langle \boldsymbol{w_i}, \boldsymbol{L}\, \boldsymbol{f}_j \rangle = \langle w_i, \boldsymbol{g} \rangle \quad i = 1,2,\ldots,N \quad (4)$$

The set of equations in Eq. 4 may be written in the matrix form as

$$\boldsymbol{ZX} = \boldsymbol{Y} \quad (5)$$

which can be solved for $\boldsymbol{Z}$ using any standard linear equation solution methodologies. The simplicity, accuracy and efficiency of the BEM lies in choosing proper set of basis/testing functions and applying to the problem at hand. In this work, we propose a special set of basis functions and a novel testing scheme to obtain accurate results.

III. Mathematical Formulation

Let T is the scalar thermal potential satisfying the Helmholtz differential equation $\nabla^2 T + k^2 T = 0$ for the time harmonic waves present in the region exterior to the surface S of the body. Another condition that thethermal potential must satisfy is the appropriate boundary conditions on the surface S of the body.

Using the potential theory and the free space Green' s function, the scattered thermal potential T^s may be defined as

$$T^s = \int_s \sigma(\boldsymbol{p}) G(\boldsymbol{p}, \boldsymbol{q}) ds' \quad (6)$$

In the above three equation,

σ is the source density function dependent of $p$over the surface of the body,

$p$is the position vector of source points, with respect to a global co-ordinate system O.

q is the position vector of observation points, with respect to a global co-ordinate system O.

$G(\boldsymbol{p}, \boldsymbol{q})$ is the free space Green's function.

$$G(\boldsymbol{p}, \boldsymbol{q}) = \frac{e^{k|p-q|} - 1}{|p-q|}$$

For a body, that has the fixed temperature defined on the surface of the body, the total thermal potential is zero, i.e.

$$\Phi^i + \Phi^s = 0 \quad (7)$$

Hence,

$$\int_s \sigma(\boldsymbol{p}) G(\boldsymbol{p}, \boldsymbol{q}) ds' = -T^i \quad (8)$$

IV. Numerical Solution Procedure

The numerical solution procedure to solve the Eq. 3.Using the basis functions is explained below.

Testing Eq. 13 with a testing function w_m, results in

$$\left\langle w_q, \int_s \sigma(\boldsymbol{p})\, G(\boldsymbol{p}, \boldsymbol{q}) ds' \right\rangle = \left\langle w_q, T^i \right\rangle \quad (9)$$

Using the inner product definition, 3, Eq. 4 can be written as

$$\int_s w_q \int_s \sigma(\boldsymbol{p})\, G(\boldsymbol{p}, \boldsymbol{q}) ds'\, ds = \int_s w_q T^i ds \quad (10)$$

The weighing function can be defined as

$$w_j = \begin{cases} 1 & q_j \in S \\ 0 & otherwise \end{cases} \quad (11)$$

Let

$$\sigma(p) = \sum_{n=1}^{N} \beta_j \boldsymbol{f}_j \quad (12)$$

Approximating the Eq. 5 over the source triangular patch,

$$\int_s w_q \int_s \sum_{j=1}^{N} \beta_j \boldsymbol{f}_j G(\boldsymbol{p_c}, \boldsymbol{q}) ds'\, ds = \int_s w_q T^i ds \quad (13)$$

Approximating the integration over the field triangular patch at the centroids,Eq. 5 becomes

$$\int_s w_i \int_s \sum_{j=1}^{N} \beta_j \boldsymbol{f}_j G(\boldsymbol{p_c}, \boldsymbol{q}_c) ds'\, ds = \int_s w_i T^i ds \quad (14)$$

Let the basis function be defined as

$$f_j = \begin{cases} 1 & p_j \in S \\ 0 & otherwise \end{cases} \quad (15)$$

$$A_i \sum_{j=1}^{N} \beta_j \boldsymbol{f}_j\, G(\boldsymbol{p_c}, \boldsymbol{q_c}) ds' = A_i\, T^i \quad (16)$$

where A_i is the area of field triangular patch. This approximation is justified because the domains are sufficiently small, which is a necessary condition to obtain accurate solution using BEM. For a pulse function defined on the source triangular patch, it results in a system of linear equations, which can be represented in the matrix form as

$$\boldsymbol{Z}\ \boldsymbol{X} = \boldsymbol{Y} \quad (17)$$

where $\boldsymbol{Z}$ is the impedance matrix of the single layer formulation of size $N_f\ X\ N_f$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are the column vectors of size N_f.The elements of $\boldsymbol{Z}$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are given below.

$$\boldsymbol{Z}_{m,n} = \int_s G(p_c, q_c) ds' \quad (18)$$

and

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$$Y_m = T^i \qquad (19)$$

where q_c is the position vector to the centroid of the field triangular patch, p_c is the position vector to the centroid of source triangular patch.Once the matrix Z is determined and vector Y is calculated, vector X can be calculated using any standard linear equation solvers.

V. Numerical Results

The solution procedure mentioned in Sec. 2 followed to obtain the temperature distribution outside a sphere of radius 1m. The surface of the sphere is maintained at a temperature of 100^{o}C. The sphere is discretized with triangular patch modeling with 4 different mesh sizes.Outer surface of the sphere is divided into 10 parts each in the polar and azimuthal directions resulting in a total of 180 triangular patches. Similarly by dividing the sphere 12, 15 and 20 parts in the polar and azimuthal directions, it results in a total of 264, 420 and 760triangular patches respectively. Higher the number of patches, the more accurate the solution is.

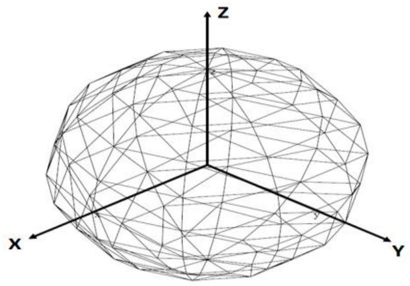

Figure 1. Triangular Patch Model of a Sphere

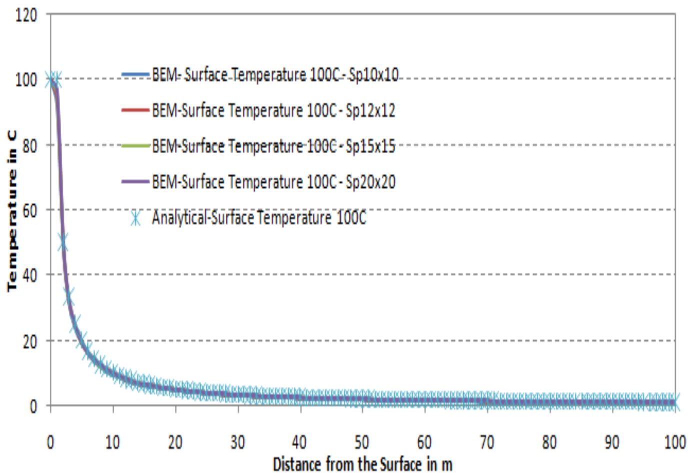

Figure 2: Temperature Distribution Outside a Sphere of Radius 1m, Maintained at a Temperature of 100^{0}C upto a Distance of 100m.

Fig. 2 shows the temperature distribution outside a sphere of radius 1m, maintained at a temperature of 100C upto a distance of 100m. The heat conduction problem considered here is a steady state model and hence it is independent of the thermal conductivity of the surrounding medium. The temperature falls very rapidly in the close vicinity of the hot surface and there after it changes very slowly and gradually as the distance increases.

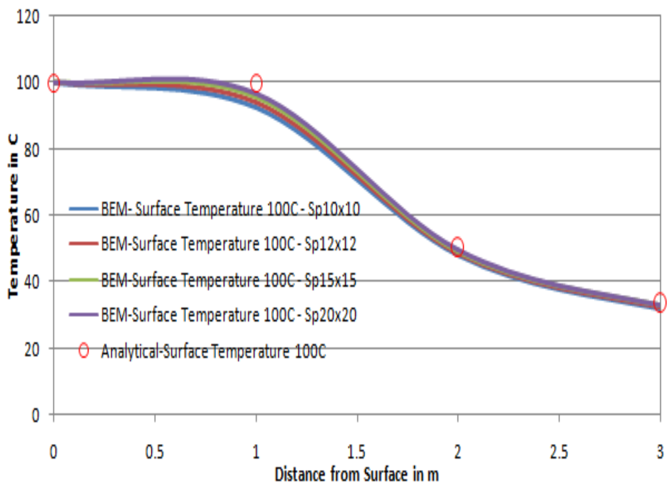

Figure 3: Temperature Distribution Outside a Sphere of Radius 1m, Maintained at a Temperature of 100C Upto a Distance of 3m.

Fig. 3 shows the temperature distribution outside a sphere of radius 1m, maintained at a temperature of 100^{0}C upto a distance of 3m. Its shows that the temperature falls by 50% at a distance of the 2m from the surface of the body. That is at distance of twice the radius of the sphere the temperature fall by 50%. The numerical results improve and converge towards the exact solution as the number of triangular patches increase from 180, 264, 420 and to 760. The exact solution for the temperature outside the sphere is given by Tb/r. Where Tb is the temperature of the body and r is the distance from the surface of the sphere.

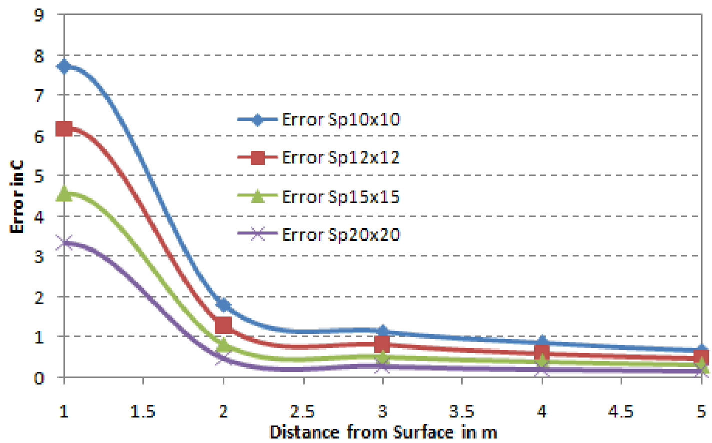

Figure 4: Temperature Difference w.r.t Exact Solution Outside a Sphere of Radius 1m, Maintained at a Temperature of 100^{0}C up to a Distance of 5m.

Fig. 4 shows the temperature difference outside a sphere of radius 1m, maintained at a temperature of 100^{0}C up to a distance of 5m. For the sphere with 120 patches (sp10x10) the error in the temperature is around 7.5^{0}C at a distance of 1m. But it comes down to 3.2^{0}C as the number patches

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increased to 760 patches. Similarly, gap between the predicted numerical results and the exact solution reduces at the other points also as the number of patches on the model increases. The solution can further be converged towards the exact solution by increasing the number of patches beyond 760 patches. However it needs more computational resources for the solution.

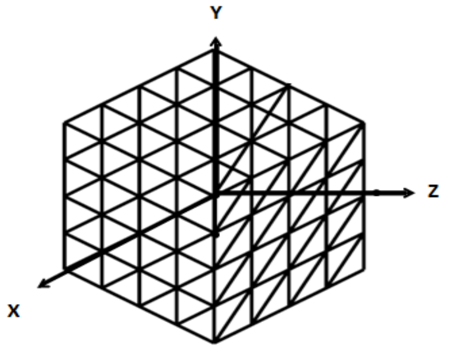

Figure 5. Triangular Patch Model of aCube

Fig. 5 shows the triangular patch modeling of the cube. Cube is divided into the triangular patches by dividing each edge into 4 parts there by creating a total of 192 patches. The surface of the cube is maintained at a temperature of 100C.temperature is predicted using the procedure mentioned above.

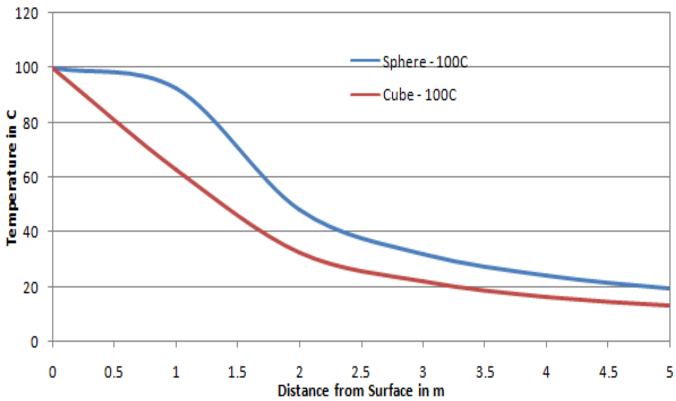

Figure 6: Temperature Distribution Outside a Cube Having Each Side of 1m, Maintained at a Temperature of 100^0C upto a Distance of 5m.

Fig. 6 shows the temperature distribution along the X-Axis up to a distance of the 5m. The temperature distribution of the cube is compared with that of the sphere. Sphere shows higher temperature since there is contribution of the bigger surface area from the sphere than that in the cube.

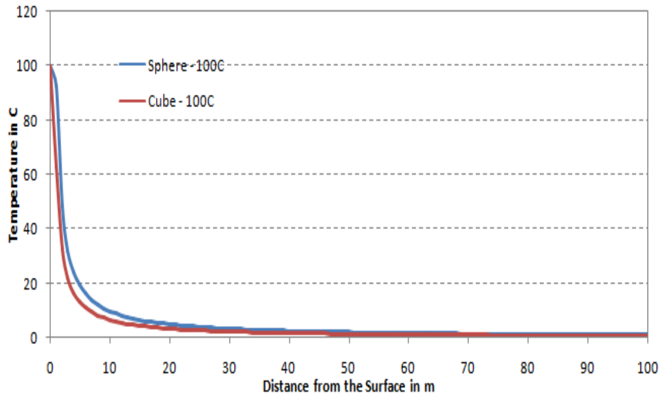

Figure 7: Temperature Distribution Outside a Cube Having Each Side of 1m, Maintained at a Temperature of 100^0C upto a Distance of 10m

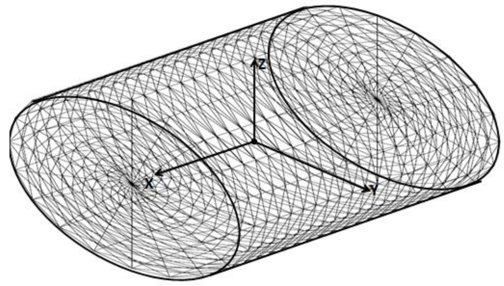

Figure 8: Triangular patch model of a Cylinder

Fig. 8 shows the triangular patch modeling of the cylinder. Cylinder is of radius 1m and height of 2m. The circumference of the cylinder is divided into 20 equal parts and height also into 20 equal parts. The cylinder is closed with two end caps. The radius of the caps are divided into 10 equal parts. This results into a total of 1560 triangular patches.The surface of the cylinder is maintained at a temperature of 100C. Temperature is predicted using the procedure mentioned above.

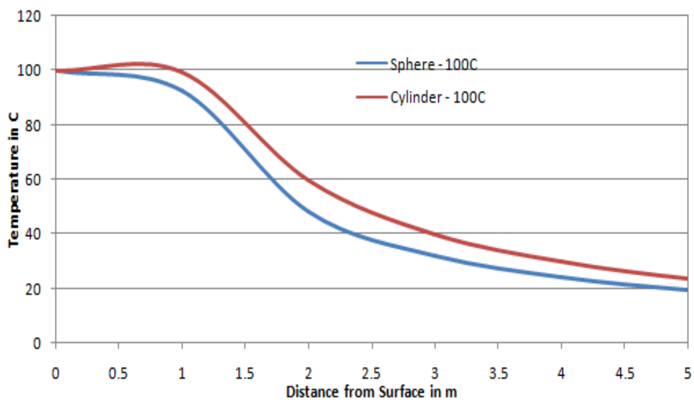

Figure 9: Temperature Distribution Outside a Cylinder Having Radius of 1m and Height 2m, Maintained at a Temperature of 100^0C upto a Distance of 5m.

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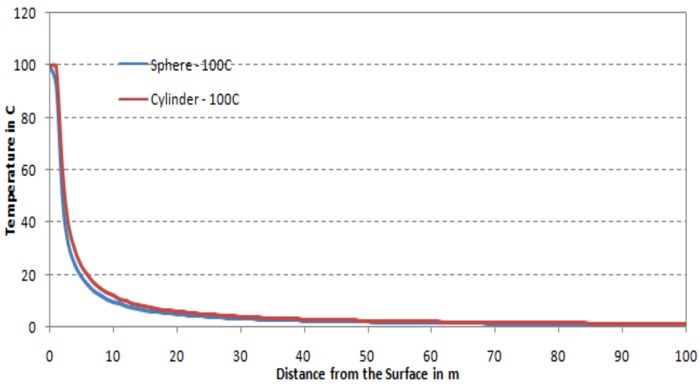

Figure 10: Temperature Distribution Outside a Cylinder Having Radius of 1m and Height 2m, Maintained at a Temperature of 100^0C upto a Distance of 100m.

Figs. 9 and 10 show the temperature distribution along the X-axis for distances up to 5m and 100m respectively. Temperature distribution resulting from the cylinder is greater than that of the sphere. This is due to the reason that, the size of the cylinder, with higher surface area, is bigger than that of the sphere. Hence there is more heat contributed from the cylinder than the sphere.

These are some examples run to demonstrate the capability of the patch based basis function. The examples are chosen to test the capability of the patch based basis function having rounded surfaces, sharp edges, sharp corners etc.

VII. Conclusions

In this work, a novel numerical method is presented to solve the thermal conduction problem using BEM methodology. The basis functions are defined on the triangular face of the patch rather than on the nodes. The solution is obtained for 4 models of the sphere and the results are compared with the exact solution. The solution converges towards the exact solution and the error narrows down from 7.5C to 3.2C as the number patches increased from 120 to 760. The exact solution is available only for the spherical geometry and hence it is possible to compare the numerical results with that of the exact solution. For geometries which are non-spherical the exact solutions are not available and hence numerical solution procedure which is validated for the spherical case can be applied. Geometries of cube having sharp edges and sharp corner, cylinder with circular edge, cone with circular edge and a sharp corner are solved for temperature distribution. The temperature of the cylinder is higher than any geometry because of the bigger size and larger surface area. It is not the intention of this research work to analyze the temperature distributions from different geometries, but only to show the capability of the patch based basis functions.

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