Structure and Study of Elements in Ternary Γ-Semigroups

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Abstract: In this paper we introduce the notion of a ternary Γ-semigroup and some examples are given. Further the terms commutative ternary Γ-semigroup, quasi commutative ternary Γ-semigroup, normal ternary Γ-semigroup, left pseudo commutative ternary Γ-semigroup, right pseudo commutative ternary Γ-semigroup are introduced and characterized them.

In section 2, the terms; ternary Γ-subsemigroup, ternary Γ-subsemigroup generated by a subset, cyclic ternary Γ-subsemigroup of a ternary Γ-semigroup and cyclic ternary Γ-semigroup are introduced and characterized them.

In section 3, we discussed some special elements of a ternary Γ-semigroups and characterized them.

INTRODUCTION

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, and the like. The theory of ternary algebraic systems was introduced by Lehmer in 1932, but earlier such structures was studied by Kasner who give the idea of n-ary algebras. Lehmer investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. Ternary semigroups are universal algebras with one associative ternary operation. The notion of ternary semigroup was known to Banach who is credited with example of a ternary semigroup which can not reduce to a semigroup. A. Anjaneyulu [i] introduced the study of pseudo symmetric ideals in semigroups. D. Madhusudhana Rao and A. Anjaneyulu [ii, iii] studied about Γ-semigroups. Further D. Madhusudhana Rao and A. Anjaneyulu and Y. Sarla [x] extended the same results to ternary semigroups. Madhusudhana Rao and Srinivasa Rao [v, vi, vii] studied about ternary semirings. In this paper mainly we extended the same results to ternary Γ-semigroups.

1: TERNARY Γ-SEMIGROUPS

We now introduce the notion of ternary Γ-semigroup

Definition 1.1: Let T and Γ be two non-empty set. Then T is said to be a Ternary Γ-semigroup if there exist a mapping from $T \times T \times T \times T \times T$ to T which maps $(x_1, x_2, x_3, x_4, x_5) \rightarrow [x_1 \alpha x_2 \beta x_3]$ satisfying the condition:

$[[x_1 \alpha x_2 \beta x_3] y x_4 x_5] = [x_1 \alpha [x_2 \beta x_3 y] x_5]$

$= [x_1 \alpha x_2 \beta [x_3 y x_4] x_5]$

$\forall x_i \in T, 1 \leq i \leq 5$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Note 1.2: For the convenience we write $x_1 \alpha x_2 \beta x_3$ instead of $[[x_1 \alpha x_2 \beta x_3] y x_4 x_5]$. Let T be a ternary Γ-semigroup. If A, B and C are three subsets of T, then we denote the set $\Gamma$ by $\{a \alpha b \beta c : a \in A, b \in B, c \in C, \alpha, \beta \in \Gamma\}$.

Note 1.3: If A, B and C are three subsets of T, then T is a ternary Γ-semigroup.

Example 1.5: Let $T = \{0, 0\}$ and $\alpha, \beta \in \Gamma$. Then $T$ is a ternary Γ-semigroup under matrix multiplication.

Example 1.6: The set $Z$ of all integers and $\alpha, \beta \in \Gamma$, then $T$ is a ternary Γ-semigroup.

Example 1.7: Let the set $Z$ of all negative integers and $\alpha, \beta \in \Gamma$, then $T$ is a ternary Γ-semigroup.

Example 1.8: Let $T = \{5n + 4 : n$ is a positive integer} and $\alpha, \beta \in \Gamma$. Then $T$ is a ternary Γ-semigroup with the operation defined by $a \alpha b \beta c = a + \alpha + b + \beta + c$ where $a, b, c \in S, \alpha, \beta \in \Gamma$ and $+$ is the usual addition of integers.

Example 1.9: Let $T = \{4n + 3 : n$ is a positive integer} and $\alpha, \beta \in \Gamma$. Then $T$ is a ternary Γ-semigroup with the operation defined by $a \alpha b \beta c = a + \alpha + b + \beta + c$ where $a, b, c \in S, \alpha, \beta \in \Gamma$ and $+$ is the usual addition of integers.

Example 1.10: Let $T = \{a, b, c\}$ and $\Gamma$ such that $a \alpha b \beta c = (x+y) \alpha z$ for all $x, y, z \in T$ and $\alpha, \beta \in \Gamma$ where $\alpha$ is defined by the table

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Then $T$ is a ternary Γ-semigroup.
Example 1.11: Let T be the set of real numbers, 0 ∈ T such that |T| > 3 and Γ be the any non empty set. Then T with the ternary operation defined by xαyβz = x if x = y = z and xαyβz = 0 otherwise is a ternary Γ-semigroup.

Example 1.12: T = {i, −i} and T = Γ. Then T is a ternary Γ-semigroup under the complex ternary operation (multiplication of complex numbers).

Example 1.13: T = {i, 0, −i} and T = Γ. Then T is a ternary Γ-semigroup under the complex ternary operation.

Example 1.14: T = {1, 2, n, −1, 0, i, 2i, …} and Γ = {−i, 0, i). Then T is a ternary Γ-semigroup under the complex ternary operation.

Example 1.15: T = {2x | x ∈ N} and Γ = N. Then T is a ternary Γ-semigroup under the ternary operation defined by [aαbβc] = H.C.F of a, b and c.

Example 1.16: Let T = Z × Z = {(a, b) : a, b ∈ Z, set of all even integers} and Γ be the any non-empty set. Then T is a ternary semigroup w. r. t the ternary multiplication defined as follows: (a, b)c(d, e) = (f, g) = (a, f).

Example 1.17: Let T = {0, {a}, {b}, {c}, {a, b}, {c, b}, {a, c}, {b, c}} and Γ = {0, {a}, {b}, {c}}. If for all A, C, E ∈ T and B, D ∈ Γ, ABCDE = A ∪ B ∩ C ∩ D ∩ E, then T is a ternary Γ-semigroup.

Example 1.18: Let T = {0, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}, {b, c}} and Γ = {{a, b, c}}. If for all A, C ∈ T and B ∈ Γ, ABCDE = A ∪ B ∩ C ∩ D ∩ E, then T is a ternary Γ-semigroup.

Example 1.19: Let T be the set of all 2 × 3 matrices over Q, the set of rational numbers and Γ be the any non empty set. Then T with the ternary operation defined from T × Γ × T × Γ × T can show that (T, .) is a ternary semigroup and we denote this set of rational numbers and Γ be the set of all 3 × 2 matrices over {Q}.

Example 1.20: Let T be a ternary Γ-semigroup and α a fixed element in Γ. We define a, b, c = aαbαc for all a, b, c ∈ T. We can show that (T, .) is a ternary semigroup and we denote this ternary semigroup by Tα.

Note 1.21: Every ternary semigroup can be considered to be a ternary Γ-semigroup. Thus the class of all ternary Γ-semigroup includes the class of all ternary semigroups.

Example 1.22 (FREE TERNARY Γ-SEMIGROUP): Let X and Γ be two nonempty sets. A sequence of elements a1a2a3a4a5 a6a7a8a9a10 where a1, a2, a3, a4, a5, a6, a7, a8, a9, a10 ∈ Γ is called a word over the alphabet X relative to Γ. The set T of all words with the operation defined from T × Γ × T to T as (a1a2a3a4a5 a6a7a8a9a10)γ = (b1b2b3b4b5 b6b7b8b9b10) γ(b1b2b3b4b5 b6b7b8b9b10) such that a1a2a3a4a5 a6a7a8a9a10 = aγ b1b2b3b4b5 b6b7b8b9b10 is a ternary Γ-semigroup. This ternary Γ-semigroup is called free ternary Γ-semigroup over the alphabet X relative to Γ.

In the following we introduce the notion of a commutative ternary Γ-semigroup.

Definition 1.23: A ternary Γ-semigroup T is said to be commutative provided a1b1c = b1c1a = c1a1b = b1a1c = c1b1a = a1c1b for all a, b, c ∈ T.
Example 1.36: Let $T = \{a, b, c, d, e\}$ and $\Gamma = \{\alpha\}$. Define a ternary operation $[\cdot]$ on $T$ as $[abc] = a \cdot b \cdot c$ where the binary operation $'\cdot'$ is defined as:

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It is easy to see that $T$ is a ternary $\Gamma$-semigroup. Now $T$ is a left pseudo commutative ternary $\Gamma$-semigroup. But $T$ is not a commutative ternary $\Gamma$-semigroup.

In the following we are introducing the notion of left pseudo commutative ternary $\Gamma$-semigroup.

**Definition 1.37**: A ternary $\Gamma$-semigroup $T$ is said to be a **left pseudo commutative** ternary $\Gamma$-semigroup provided $\forall a, b, c \in T$,

\[ a(c\Gamma d)\Gamma e = a(c\Gamma b\Gamma c)\Gamma e = a(c\Gamma b\Gamma d)\Gamma e = a(c\Gamma b\Gamma c\Gamma d)\Gamma e = a(c\Gamma b\Gamma d\Gamma c)\Gamma e = a(c\Gamma b\Gamma d\Gamma c\Gamma e) \]

for all $a, b, c, d, e \in T$.

**Theorem 1.38**: If $T$ is a commutative ternary semigroup then $T$ is a lateral pseudo commutative ternary $\Gamma$-semigroup.

**Proof**: Suppose that $T$ is a commutative ternary $\Gamma$-semigroup. Then $a(c\Gamma b)\Gamma d\Gamma e = a(c\Gamma b)\Gamma d\Gamma e = a(c\Gamma b)\Gamma d\Gamma e = a(c\Gamma b)\Gamma d\Gamma e = a(c\Gamma b)\Gamma d\Gamma e = a(c\Gamma b)\Gamma d\Gamma e = a(c\Gamma b)\Gamma d\Gamma e$

for all $a, b, c, d, e \in T$.

**Example 1.40**: Consider the ternary $\Gamma$-semigroup in example 1.36, $T$ is a lateral pseudo commutative. But $T$ is not a commutative ternary $\Gamma$-semigroup.

In the following we are introducing the notion of right pseudo commutative ternary $\Gamma$-semigroup.

**Definition 1.41**: A ternary $\Gamma$-semigroup $T$ is said to be a **right pseudo commutative** provided $\forall a, b, c \in T$,

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a, b, c ∈ S₂, α, γ ∈ Γ, S₂ is a ternary Γ-subsemigroup of T
⇒ abγc ∈ S₂

abγc ∈ S₁, abγc ∈ S₂ ⇒ abγc ∈ S₁ \cap S₂.

Therefore S₁ \cap S₂ is a ternary Γ-subsemigroup of T.

**Theorem 2.7:** The non-empty intersection of any family of ternary Γ-subsemigroups of a ternary Γ-semigroup T is a ternary Γ-subsemigroup of T.

**Proof:** Straight forward.

In the following we are introducing a ternary Γ-subsemigroup which is generated by a subset and a cyclic ternary Γ-subsemigroup of ternary Γ-semigroup.

**Definition 2.8:** Let T be a ternary Γ-semigroup and A be a non-empty subset of T. The smallest ternary Γ-subsemigroup of T containing A is called a ternary Γ-subsemigroup of T generated by A. It is denoted by (A).

**Theorem 2.9:** Let T be a ternary Γ-semigroup and A be a non-empty subset of T. Then (A) = \{ a₁a₂a₃..., aₙaₙ⁺₁ | aₙ⁺₁ ∈ A, a₁, a₂, ..., aₙ ∈ T \}

**Proof:** Let S = \{ a₁a₂a₃..., aₙaₙ⁺₁ | aₙ⁺₁ ∈ A, a₁, a₂, ..., aₙ ∈ A \}. Let a, b, c ∈ S and α, γ ∈ Γ.

a ∈ T ⇒ a₁ = a₁, a₂, a₃,..., aₙ = aₙ ∈ T. Let a₁, a₂, ..., aₙ⁺₁ ∈ \Gamma.

b ∈ T ⇒ b₁ = b₁, b₂, b₃,..., bₙ⁺₁ = bₙ⁺₁ ∈ \Gamma.

Let c ∈ S ⇒ c = c₁c₂c₃...cₙ⁻¹cᵣ ∈ A, γ₁, γ₂,..., γₙ⁻¹ ∈ \Gamma.

Now abγc = (a₁a₂a₃...aₙaₙ⁺₁γ) (b₁b₂b₃...bₙ⁺₁) (c₁c₂c₃...cₙ⁻¹cᵣ) ∈ S.

Therefore S is a ternary Γ-subsemigroup of T.

Let K be a ternary Γ-subsemigroup of S such that A ⊆ K.

Let a ∈ K. Then a = a₁a₂a₃...aₙaₙ⁺₁ where a₁, a₂, a₃,..., aₙ⁺₁ ∈ A, a₁, a₂, ..., aₙ⁺₁ ∈ \Gamma.

a₁, a₂, ..., aₙ⁺₁ ∈ K, a₁, a₂, ..., aₙ⁺₁ ∈ \Gamma, K is a ternary Γ-subsemigroup \Rightarrow a₁a₂a₃...aₙaₙ⁺₁ ∈ K ⇒ a ∈ K. Therefore SC ⊆ K.

So S is the smallest ternary Γ-subsemigroup of T containing A. Hence (A) = S.

**Theorem 2.10:** Let T be a ternary Γ-semigroup and A be a non-empty subset of T. Then (A) is the intersection of all ternary Γ-subsemigroups of T containing A.

**Proof:** Let Δ be the set of all ternary Γ-subsemigroups of T containing A. Since T is a ternary Γ-subsemigroup of T containing A, T ∈ Δ, so Δ ≠ ∅.

Let S⁺ = \bigcap_{α∈Δ} S. Since A ⊆ S for all S ∈ Δ and A ⊆ S⁺

By theorem 2.7, S⁺ is a ternary Γ-subsemigroup of T.

Since S⁺ ⊆ S for all S ∈ Δ, S⁺ is the smallest ternary Γ-subsemigroup of T containing A. Therefore S⁺ = (A).

**Definition 2.11:** Let T be a ternary Γ-semigroup. A ternary Γ-subsemigroup S of T is said to be a cyclic ternary Γ-subsemigroup of T if S is generated by a single element subset of T.

**Definition 2.12:** A ternary semigroup T is said to be a cyclic ternary semigroup if T is cyclic ternary subsemigroup of T itself.

**3. SPECIAL ELEMENTS OF A TERNARY Γ-SEMIGROUP**

In the following we are introducing left identity, lateral identity, right identity, two sided identity and identity of ternary Γ-semigroup.

**Definition 3.1:** An element a of ternary Γ-semigroup T is said to be left identity of T provided aαβt = t for all t ∈ T, α, β ∈ Γ.

**Note 3.1.2:** Left identity element a of a ternary Γ-semigroup T is also called as left unital element.

**Definition 3.3:** An element a of a ternary Γ-semigroup T is said to be a lateral identity of T provided aαtβ = t for all t ∈ T, α, β ∈ Γ.

**Note 3.3:** Lateral identity element a of a ternary Γ-semigroup T is also called as lateral unital element.

**Definition 3.7:** An element a of a ternary Γ-semigroup T is said to be a two sided identity of T provided aαβt = aαtβ = t for all t ∈ T, α, β ∈ Γ.

**Note 3.8:** Two-sided identity element a of a ternary Γ-semigroup T is also called as bi-unital element.

**Definition 3.9:** An element a of a ternary Γ-semigroup T is said to be an identity provided aαβt = tαβt = tαtβ = t for all t ∈ T, α, β ∈ Γ.

**Note 3.10:** An identity element a of a ternary Γ-semigroup T is also called as unital element.

**Note 3.11:** An element a of a ternary Γ-semigroup T is an identity of T iff a is left identity, lateral identity and right identity of T.

**Example 3.2:** Let Z₀⁻ be the set of all non-positive integers and Γ be the set of binary operations. Then with the usual ternary multiplication, Z₀⁻ forms a ternary Γ-semigroup with identity element -1.

**Note 3.13:** The identity ( if exists ) of a ternary Γ-semigroup is usually denoted by 1 ( or ) e.

**Definition 3.14:** A ternary Γ-semigroup T with identity is called a ternary Γ-monoid.

**Notation 3.15:** Let T be a ternary Γ-semigroup. If T has an identity, let T¹ = T and if T does not have an identity, let T¹ be the ternary semigroup T with an identity adjoining usually denoted by the symbol 1.

In the following we are introducing left zero, lateral zero, right zero, two sided zero and zero of ternary Γ-semigroup.

**Definition 3.16:** An element a of a ternary Γ-semigroup T is said to be a left zero of T provided aαβc = a for all c ∈ T, α, β ∈ Γ.

**Definition 3.17:** An element a of a ternary Γ-semigroup T is said to be a lateral zero of T provided βαc = c for all c ∈ T, α, β ∈ Γ.
Definition 3.18: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be a right zero of $T$ provided $bac = a \forall b, c \in T$, $a, b \in \Gamma$. 

Definition 3.19: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be a two sided zero of $T$ provided $a\beta\gamma = \alpha\beta\gamma = a \forall b, c \in T$, $a, \beta, \gamma \in \Gamma$.

Note 3.20: If $a$ is a two sided zero of a ternary $\Gamma$-semigroup $T$, then $a$ is both left zero and right zero of $T$.

Definition 3.21: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be zero of $T$ provided $a\beta\gamma = b\alpha\beta = c\beta\alpha = a \forall b, c \in T$, $a, \beta, \gamma \in \Gamma$.

Note 3.22: If $a$ is a zero of $T$, then $a$ is a left zero, lateral zero and right zero of $T$.

Theorem 3.23: If $a$ is a left zero, $b$ is a lateral zero and $c$ is a right zero of a ternary $\Gamma$-semigroup $T$, then $a = b = c$.

Proof: Since $a$ is a left zero of $T$, $a\beta\gamma = a$ for all $a, b, c \in T$. Since $b$ is a lateral zero of $T$, $a\beta\gamma = b$. Since $c$ is a right zero of $T$, $a\beta\gamma = c$. Therefore $a\beta\gamma = a = b = c$.

Theorem 3.24: Any ternary $\Gamma$-semigroup has at most one zero element.

Proof: Let $a, b, c$ be three zeros of a ternary $\Gamma$-semigroup $T$. Now $a$ can be considered as a left zero, $b$ can be considered as a lateral zero and $c$ can be considered as a right zero of $T$. By Theorem 3.24, $a = b = c$. Then $T$ has at most one zero.

Note 3.25: The zero (if exists) of a ternary $\Gamma$-semigroup is usually denoted by 0.

Notation 3.26: Let $T$ be a ternary $\Gamma$-semigroup. If $T$ has a zero, let $T^0 = T$ and if $T$ does not have a zero, let $T^0$ be the ternary $\Gamma$-semigroup $T$ with zero adjoined usually denoted by the symbol 0.

In the following we introducing the notion of left zero ternary $\Gamma$-semigroup, lateral zero ternary $\Gamma$-semigroup, right zero ternary $\Gamma$-semigroup and zero ternary $\Gamma$-semigroup.

Definition 327: A ternary $\Gamma$-semigroup in which every element is a left zero is called a left zero ternary $\Gamma$-semigroup.

Definition 3.28: A ternary $\Gamma$-semigroup in which every element is a lateral zero is called a lateral zero ternary $\Gamma$-semigroup.

Definition 3.29: A ternary $\Gamma$-semigroup in which every element is a right zero is called a right zero ternary $\Gamma$-semigroup.

Definition 3.30: A ternary $\Gamma$-semigroup with 0 in which the product of any three elements equal to 0 is called a zero ternary $\Gamma$-semigroup (or) null ternary $\Gamma$-semigroup.

Example 3.31: Let $0 \in T \subseteq R$ and $|T| > 2$ and $\Gamma$ be the any non-empty set. Then $T$ with the ternary operation defined by $xy\beta\gamma = x$ if $x = y = z$ and $xy\beta\gamma = 0$ otherwise is a ternary $\Gamma$-semigroup with zero.

In the following we are introducing the notion of idempotent element of a ternary $\Gamma$-semigroup.

Definition 3.32: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be an $\alpha$-idempotent element provided $\alpha\alpha\alpha\alpha\alpha = a$.

Note 3.33: The set of all idempotent elements in a ternary $\Gamma$-semigroup $T$ is denoted by $E_T(T)$.

Example 3.34: Every identity, zero elements are $\alpha$-idempotent elements.

Definition 3.35: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be an $(\alpha, \beta)$-idempotent element provided $\alpha\alpha\beta\beta\alpha = a$ for all $\alpha, \beta \in \Gamma$.

Note 3.36: In a ternary $\Gamma$-semigroup $T$, $a$ is an idempotent if $a$ is an $(\alpha, \beta)$-idempotent for all $\alpha, \beta \in \Gamma$.

Note 3.37: If an element $a$ of a ternary $\Gamma$-semigroup $T$ is an idempotent, then $a\Gamma\alpha\Gamma\alpha = a$.

In the following we introduce proper idempotent element ternary $\Gamma$-semigroup.

Definition 3.38: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be a proper idempotent element provided $a$ is an idempotent which is not the identity of $T$ if identity exists.

We now introduce an idempotent ternary $\Gamma$-semigroup and a strongly idempotent ternary $\Gamma$-semigroup.

Definition 3.39: A ternary $\Gamma$-semigroup $T$ is said to be an idempotent ternary $\Gamma$-semigroup provided every element of $S$ is an idempotent for some $\alpha \in \Gamma$.

Definition 3.40: A ternary $\Gamma$-semigroup $T$ is said to be a strongly idempotent ternary $\Gamma$-semigroup or ternary $\Gamma$-band provided every element in $T$ is an idempotent for some $\alpha \in \Gamma$.

In the following we are introducing regular element and regular ternary $\Gamma$-semigroup.

Definition 3.41: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be $\alpha$-regular if there exist $x, y \in T$ and $\alpha, \beta, \gamma \in \Gamma$ such that $\alpha\alpha\gamma\beta\gamma\alpha = a$.

Definition 3.42: A ternary $\Gamma$-semigroup $T$ is said to be regular ternary $\Gamma$-semigroup provided every element is regular.

Example 3.43: Let $T = \{0, a, b\}$ and $\Gamma$ be any nonempty set. If we define a binary operation on $T$ as the following Cayley table, then $T$ is a ternary $\Gamma$-semigroup.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
</tbody>
</table>

Define a mapping from $T \times \Gamma \times T \times \Gamma \times T \times T$ to $T$ as $a\alpha\beta\gamma\delta\beta = abc$ for all $a, b, c \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Then $T$ is a regular ternary $\Gamma$-semigroup.

Theorem 3.44: Every $\alpha$-idempotent element in a ternary $\Gamma$-semigroup is regular.

Proof: Let $a$ be an $\alpha$-idempotent element in a ternary $\Gamma$-semigroup $T$. Then $a = \alpha\alpha\alpha\alpha\alpha\alpha = \alpha\alpha\alpha\alpha\alpha\alpha = \alpha\alpha\alpha\alpha\alpha\alpha$. Therefore $a$ is regular element.

In the following we are introducing the notion of left regular, lateral regular right regular, intra regular and completely regular elements of a ternary $\Gamma$-semigroup and completely regular ternary $\Gamma$-semigroup.

Definition 3.45: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be left regular if there exist $x, y \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = \alpha\alpha\beta\gamma\delta\beta$. i.e., $a \in a\Gamma\alpha\Gamma\alpha\Gamma\alpha\Gamma$.

Definition 3.46: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be lateral regular if there exist $x, y \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = \alpha\beta\alpha\gamma\delta\gamma\gamma\beta\gamma$. i.e., $a \in T\Gamma\alpha\Gamma\alpha\Gamma\alpha\Gamma$.

Definition 3.47: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be right regular if there exist $x, y \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = \alpha\beta\gamma\alpha\beta\gamma\alpha\beta\gamma$. i.e., $a \in T\Gamma\Gamma\alpha\Gamma\alpha\Gamma\alpha\Gamma$.
Definition 3.48: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be **completely regular** if there exist $x, y \in T$ such that $a = axb = bxa = c$ and $a = ayb = bya = c$. Note 3.49: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be **completely regular** if there exist $x, y \in T$ and $a, b, c \in T$ such that $a = axb = bxa = c$ and $a = ayb = bya = c$.

Definition 3.50: A ternary $\Gamma$-semigroup $T$ is said to be a **completely regular ternary $\Gamma$-semigroup** provided every element in $T$ is completely regular.

Definition 3.51: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be **intra regular** if there exist $x, y \in T$ and $a, b, c \in T$ such that $a = axb = bxa = c$.

Theorem 3.52: Let $T$ be a ternary $\Gamma$-semigroup and $a \in T$. If $a$ is a completely regular element, then $a$ is regular, left regular, lateral regular and right regular.

**Proof:** Suppose that $a$ is completely regular.

Then there exist $x, y \in T$ and for all $a, b, c \in T$ such that $a = axb = bxa = c$ and $a = ayb = bya = c$. Clearly $a$ is regular.

Now $a = axb = bxa = c$. Therefore $a$ is left regular.

Also $a = axb = bxa = c$. Therefore $a$ is lateral regular.

and $a = axb = bxa = c$. Therefore $a$ is right regular.

In the following we are introducing the notion of mid unit of a ternary $\Gamma$-semigroup.

Definition 3.53: An element $a$ of a ternary $\Gamma$-semigroup $T$ is said to be a **mid-unit** provided $x\Gamma a\Gamma y\Gamma z = x\Gamma y\Gamma z$ for all $x, y, z \in T$.

CONCLUSION

D. Madhusudhana Rao and A. Anjaneyulu studied about $\Gamma$-semigroups. Further D. Madhusudhana Rao and A. Anjaneyulu and Y. Sarla extended the same results to ternary semigroups. In this paper mainly we extended the same results to ternary $\Gamma$-semigroups.

ACKNOWLEDGEMENT

The authors would like to thank the referee(s) for careful reading of the manuscript.

REFERENCES


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